Sylvie Corteel's One-Line Proof of a Partition Theorem Discovered by Andrews-Paule-Riese's Computer

In the conference 'Combinatorics and Physics' organized by Bill Chen and Jim Louck (Los Alamos, Aug. 1998), George Andrews stated an elegant computer-generated theorem that came out of his project, with Peter Paule and Axel Riese, of automating MacMahon's Omega calculus.

Theorem: The number of partitions of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, such that $\Delta^r \lambda$ has non-negative entries (here $\Delta(\mu_1, \mu_2, \ldots, \mu_k) := (\mu_1 - \mu_2, \ldots, \mu_{k-1} - \mu_k, \mu_k))$ equals the number of partitions of n into parts of the form $\binom{i}{r}$.

A few minutes after the talk, Sylvie Corteel (who was probably the youngest participant), came up with a brilliant human-generated one-line-proof, that establishes a *natural bijection* between the two sets. To wit it is:

$$\lambda \to \left\{ \binom{i-1+r}{r}^{(\Delta^r \lambda)_i} \right\}, \ (1 \leq i \leq k) \quad ,$$

where in the right side we use the frequency notation of partitions a^b meaning 'a repeated b times'. The right hand side is indeed a partition of n (by summation by parts r times), and the reverse bijection is:

$$\left\{ \binom{i-1+r}{r}^{\nu_i} \right\} (1 \leq i \leq k) \to \left(\sum_{j=i}^{k} \binom{j+r-i}{r-1} \nu_j \right)_{1 \leq i \leq k} \quad ,$$

which again follows by summation by parts (this time only $r-1$ times). Note that when $r = 1$ this is the classical bijection between a partition and its conjugate.

Reported by: Doron Zeilberger, zeilberg@math.temple.edu