

How Many Singles, Doubles, Triples, Etc., Should The Coupon Collector Expect?

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There are m equi-probable baseball cards placed at random in chewing gums. It is well known and easy to see that a collector should expect to buy $m(1+1/2+1/3+\dots+1/m)$ gums before acquiring all the kinds of cards. At the end, he would have some singles, some doubles, some triples, etc. Let $A(m, i)$ be the expected number of kinds of cards of which he has exactly i copies of. Here I give a short proof of:

Formula (Foata-Han-Lass[1]): $\sum_{i=1}^{\infty} A(m, i)t^i = t - 1 + m! / \prod_{j=2}^m (j - t)$.

Proof: Let's number the (kinds of) cards, in order of first arrival by $1, 2, \dots, m$. The purchased cards define a word given by the regular expression $1^*2\{1, 2\}^*3\{1, 2, 3\}^*4\dots\{1, 2, \dots, m-1\}^*m$, whose (probability) generating function is

$$f(x_1, \dots, x_m) = \frac{x_m}{m} \prod_{j=1}^{m-1} \frac{x_j}{m - (x_1 + \dots + x_j)} .$$

Let a *marked word* be a pair $[w, i]$ where w is a word that is an instance of the above regular expression, and $1 \leq i \leq m$. By the familiar trick of computing expectations by changing the order of summation, it follows that the left side of the Foata-Han-Lass formula is the sum of the weights of all eligible marked words, where $weight([w, i]) := (1/m)^{|w|}$ times t raised to the power [the number of times the letter i occurs in w]. For example, if $m = 3$ and $w=1111211213$, then $weight([w, 1]) = (1/3)^{10}t^7$, $weight([w, 2]) = (1/3)^{10}t^2$, $weight([w, 3]) = (1/3)^{10}t$. Hence $\sum_{i=1}^{\infty} A(m, i)t^i = m! \sum_{i=1}^m f_i$ (the factor of $m!$ is to account for all possible orderings), where f_i is f with all the x 's replaced by 1, except for x_i that is replaced by t . But

$$f_{m-i} = \frac{1}{m} \prod_{j=1}^{m-i-1} \frac{1}{m-j} \cdot \frac{t}{i+1-t} \cdot \prod_{j=m-i+1}^{m-1} \frac{1}{m-j+1-t} = \frac{i!t}{m!(2-t)(3-t)\dots(i+1-t)} ,$$

when $i > 0$ and $f_m = t/m!$. Hence

$$\sum_{i=1}^{\infty} A(m, i)t^i = t + t \sum_{i=1}^{m-1} \frac{i!}{(2-t)(3-t)\dots(i+1-t)} = t + \frac{m!}{(2-t)(3-t)\dots(m-t)} - 1 \quad \square.$$

Reference

1. D. Foata, G.-N. Han, et B. Lass, *Les nombres hyperharmoniques et la fratrie du collectionneur de vignettes*, preprint, available from Foata's website.

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<http://www.math.temple.edu/~zeilberg/> . July 5, 2001 Supported in part by the NSF.