Two Proofs that $\sum_{k=0}^{n}(-1)^k k! S(n,k) = (-1)^n$

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Proof 1: $k!S(n,k)$ is $n!$ times the coeff. of x^n in $(e^x-1)^k$, so the sum equals $n!$ times the coeff. of x^n in $\sum_{k=0}^{\infty}(-1)^k(e^x-1)^k = 1/(1+(e^x-1)) = e^{-x}$, that equals $n! \cdot (-1)^n/n! = (-1)^n$ $\square$.

Proof 2: $k!S(n,k)$ is the number of ways of placing n balls (labeled $1,\ldots,n$) into k non-empty boxes, lined-up from left to right. The identity of the title says that the number of ways of doing it with an *even* number of boxes is "almost" the same as doing it with an *odd* number of boxes, and the difference is 1 (in favor of the evens if n is even and in favor of the odds if n is odd). So we need a parity-changing involution.

If ball 1 is alone in box ℓ, where $\ell \neq 1$, move it to box $\ell - 1$ (and discard box ℓ). If 1 has box-mates, create a new box immediately to its right and put 1 there.

If 1 is alone in the first box, do the same for ball 2. If 1 is alone in the first box, and 2 is alone in the second box, do it for 3, etc.. Every arrangement can be matched except for the scenario $\{1\}, \{2\}, ..., \{n\}$, that makes the difference 1 if n is even and -1 if n is odd $\square$.

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