

**Proof of a Conjecture of Philippe di Francesco and Paul Zinn-Justin related to the qKZ equations  
and to Dave Robbins' Two Favorite Combinatorial Objects**

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For any Laurent polynomial  $P(z_1, \dots, z_n)$  in the variables  $(z_1, \dots, z_n)$ , recall that the *anti-symmetrizer*,  $AS(P)$  is defined by:

$$AS(P)(z_1, \dots, z_n) = \sum_{\pi \in S_n} (-1)^{inv(\pi)} P(z_{\pi(1)}, \dots, z_{\pi(n)}) \quad ,$$

where  $S_n$  is the group of permutations on  $\{1, \dots, n\}$  and  $inv(\pi)$  is the number of inversions of  $\pi$  (the number of pairs  $1 \leq i < j \leq n$  such that  $\pi(i) > \pi(j)$ ).

Also define the *negative-part* of  $P$ ,  $N(P)$ , to be the polynomial in  $1/z_1, \dots, 1/z_n$  obtained by removing all monomials that have at least one positive exponent.

For example,  $AS(5 + z_1^{-1} + z_1^3 z_2^{-2}) = z_1^{-1} + z_1^3 z_2^{-2} - z_2^{-1} - z_2^3 z_1^{-2}$ , and  $N(5 + z_1^{-1} + z_1^3 z_2^{-2}) = 5 + z_1^{-1}$ .

Let

$$F_n(z_1, \dots, z_n) = \prod_{i=1}^n z_i^{-2i+2} \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j) \prod_{1 \leq i < j \leq n} (1 + z_i z_j + T z_j) \quad ,$$

$$B_n(z_1, \dots, z_n) = \prod_{1 \leq i < j \leq n} (z_j^{-1} - z_i^{-1})(T + z_i^{-1} + z_j^{-1}) \quad .$$

In [1] (Eq. (4.4)), the following conjecture was made:

$$N [AS [F_n(z_1, \dots, z_n)]] = B_n(z_1, \dots, z_n) \quad . \quad (diFJZ)$$

Let  $G_n = AS(F_n)$ . By breaking-up the summation over  $S_n$  into those  $\pi$  for which  $\pi(n) = i$  ( $i = 1, \dots, n$ ),  $G_n$  is seen to satisfy the recurrence

$$G_n(z_1, \dots, z_n) = \sum_{j=1}^n (-1)^{n-j} \left( z_j^{1-2n} (1 - z_j^2) \prod_{i=1, i \neq j}^n [(1 + T z_j + L z_i z_j)(1 - z_i z_j)] \right) \cdot$$

$$G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \quad .$$

Let  $H_n = N(G_n)$  (our object of desire). Applying the  $N$ -operation to both sides, and noting that since the factor in front of  $G_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$  only has *positive* exponents in  $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ ,

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we have

$$H_n(z_1, \dots, z_n) = N \left\{ \sum_{j=1}^n (-1)^{n-j} z_j^{1-2n} (1 - z_j^2) \prod_{i=1, i \neq j}^n [(1 + Tz_j + Lz_i z_j)(1 - z_i z_j)] \cdot H_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \right\} .$$

By the inductive hypothesis  $H_{n-1} = B_{n-1}$  so we have to prove that

$$B_n(z_1, \dots, z_n) = N \left\{ \sum_{j=1}^n (-1)^{n-j} z_j^{1-2n} (1 - z_j^2) \prod_{i=1, i \neq j}^n [(1 + Tz_j + Lz_i z_j)(1 - z_i z_j)] \cdot B_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \right\} .$$

Now comes a:

**Nice Surprise:**

$$\sum_{j=1}^n \left( (-1)^{n-j} z_j^{1-2n} (1 - z_j^2) \prod_{i=1, i \neq j}^n [(1 + Tz_j + Lz_i z_j)(1 - z_i z_j)] \right) \cdot B_{n-1}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) = B_n(z_1, \dots, z_n) (1 - z_1^2 z_2^2 \cdots z_n^2) . \quad (\text{Nice Surprise})$$

By the *pigeonhole principle*, each and every monomial featuring in  $B_n(z_1, \dots, z_n) z_1^2 z_2^2 \cdots z_n^2$  has at least one positive exponent, hence

$$N(B_n(z_1, \dots, z_n) (1 - z_1^2 z_2^2 \cdots z_n^2)) = B_n(z_1, \dots, z_n) ,$$

and (*diFJZ*) would follow. It remains to prove the Nice Surprise. By dividing both sides of (*Nice Surprise*) by  $B_n$ , and rearranging terms, we see that it is equivalent to the following Intriguing Identity:

$$\sum_{j=1}^n \frac{(2z_j + T) \prod_{i=1}^n [(1 + z_i(T + z_j))(1 - z_i z_j)]}{(1 + z_j T + z_j^2) \prod_{i=1, i \neq j}^n (z_j - z_i) \prod_{i=1}^n (z_j + z_i + T)} = (-1)^{n-1} (1 - z_1^2 \cdots z_n^2) . \quad (\text{Intriguing Identity})$$

Let  $\alpha = \alpha(T)$  and  $\beta = \beta(T)$  be the two roots of  $1 + zT + z^2 = 0$ . Note that  $\alpha + \beta = -T$  and  $\alpha\beta = 1$ .

Now it is obvious whom to call for help! Dear old Count Joe!, a.k.a. as Joseph-Louis Lagrange, *Comte de l'Empire*, whose humble (and a posteriori trivial) *Lagrange Interpolation Formula* is at least as useful as Euler-Lagrange and Lagrange multipliers and sum-of-four-squares combined! To wit, for any polynomial  $P(z)$  of degree  $\leq N - 1$  and *any* numbers  $w_1, \dots, w_N$ , we have

$$P(z) = \sum_{j=1}^N \frac{P(w_j) \prod_{i=1, i \neq j}^N (z - w_i)}{\prod_{i=1, i \neq j}^N (w_j - w_i)} . \quad (LI)$$

(Proof: both sides are equal at the  $N$  numbers  $z = w_1, \dots, w_N$ , hence they *always* equal (being polynomials of degree  $\leq N - 1$ ).

In fact, we will only need the corollary obtained by comparing the coefficient of  $z^{N-1}$  on both sides

$$\text{Coeff of } z^{N-1} \text{ of } P(z) = \sum_{j=1}^N \frac{P(w_j)}{\prod_{i=1, i \neq j}^n (w_j - w_i)} . \quad (LI')$$

Now Lagrange-Interpolate

$$P(z) := (2z + T) \prod_{i=1}^n [(1 + z_i(T + z))(1 - z_i z)] ,$$

with respect to the  $2n + 2$  numbers

$$\{z_1, \dots, z_n, -z_1 - T, \dots, -z_n - T, \alpha, \beta\} ,$$

in order to get

$$\begin{aligned} 2(-1)^n z_1^2 \cdots z_n^2 &= \sum_{j=1}^n \frac{P(z_j)}{(1 + z_j T + z_j^2) \prod_{i=1, i \neq j}^n (z_j - z_i) \prod_{i=1}^n (z_j + z_i + T)} \\ &+ \sum_{j=1}^n \frac{P(-T - z_j)}{(1 + z_j T + z_j^2) \prod_{i=1, i \neq j}^n (-z_j - T - z_i) \prod_{i=1}^n (-z_j - T + z_i + T)} \\ &+ \frac{P(\alpha)}{(\alpha - \beta) \prod_{i=1}^n (\alpha - z_i) \prod_{i=1}^n (\alpha + z_i + T)} \\ &+ \frac{P(\beta)}{(\beta - \alpha) \prod_{i=1}^n (\beta - z_i) \prod_{i=1}^n (\beta + z_i + T)} . \end{aligned}$$

Since  $P(-T - z_j) = -P(z_j)$  (check!) and using  $\alpha\beta = 1$  and  $\alpha + \beta = -T$  to simplify, we get that (*IntriguingIdentity*) is indeed true.  $\square$ .

## Reference

1. P. di Francesco and P. Zinn-Justin, *Quantum Knizhnik-Zamolodchikov equation, Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices*, <http://www.arxiv.org/abs/math-ph/0703015> .