An Inelegant (but Short(!)) Proof of a Major Index Theorem of Garsia and Gessel

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Recall Adriano Garsia's favorite notation: $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false. For example $\chi(MathIsFun) = 1$, $\chi(HeleneBarceloIsAGoodEditor) = 0$. Also recall that the *major index* of any list of integers $\pi = (\pi_1, \dots, \pi_n)$ (in particular a permutation) is defined by $maj(\pi) := \sum_{i=1}^{n-1} i\chi(\pi_i > \pi_{i+1})$. Let $\theta = \theta_1 \dots \theta_a$ and $\pi = \pi_1 \dots \pi_b$ be two disjoint lists of distinct integers. Let $S(\theta, \pi)$ be the set of (a+b)!/(a!b!) mergings (suffles) of θ and π. Let $(q)_n := (1-q)(1-q^2)\cdots(1-q^n)$. Moti Novick[N] has recently found an elegant bijective proof of the following

Theorem (Garsia and Gessel[GG]):

$$\sum_{\sigma \in S(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b}}{(q)_a (q)_b} q^{maj(\theta)+maj(\pi)} \quad . \qquad (Moti)$$

In this short note I will present an *inelegant*, induction proof, by proving, more generally:

Lemma: Let $S_1(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is θ_a, and let $S_2(\theta, \pi)$ be the subset of $S(\theta, \pi)$ whose last entry is π_b, then

$$\sum_{\sigma \in S_1(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_{a-1} (q)_b} q^{maj(\theta)+maj(\pi)+b\chi(\pi_b > \theta_a)} \quad , \qquad (Adriano)$$

$$\sum_{\sigma \in S_2(\theta,\pi)} q^{maj(\sigma)} = \frac{(q)_{a+b-1}}{(q)_a (q)_{b-1}} q^{maj(\theta)+maj(\pi)+a\chi(\pi_b < \theta_a)} \quad . \qquad (Ira)$$

Note that adding-up $(Adriano)$ and (Ira) gives $(Moti)$. Let's call the left-sides of $(Adriano)$ and (Ira) $F_1(a, b)$ and $F_2(a, b)$ respectively, and let's call their right-sides $G_1(a, b)$ and $G_2(a, b)$ respectively. It is immediate that $(F_1, F_2) = (G_1, G_2)$ when $a = 0$ or $b = 0$, and the fact that $(F_1, F_2) = (G_1, G_2)$ for all $a, b \geq 0$ follows from the fact that both $(X_1, X_2) = (F_1, F_2)$ and $(X_1, X_2) = (G_1, G_2)$ satisfy the recurrence

$$X_1(a, b) = q^{(a+b-1)\chi(\theta_{a-1} > \theta_a)} X_1(a-1, b) + q^{(a+b-1)\chi(\pi_b > \theta_a)} X_2(a-1, b)$$

$$X_2(a, b) = q^{(a+b-1)\chi(\theta_a > \pi_b)} X_1(a, b-1) + q^{(a+b-1)\chi(\pi_{b-1} > \pi_b)} X_2(a, b-1) \quad .$$

Remarks: **1.** Moti Novick's elegant bijection also preserves equidistribution over Inverse Descent Classes. **2.** It would be interesting to bijectify the above inductive proof, along the lines of [MZ], and see if the resulting bijection is identitical, or similar, to Moti Novick's elegant bijection.

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`http://www.math.rutgers.edu/~zeilberg/` . First version: March 25, 2009. This version: March 29, 2009 (Thanks to Moti Novick). Supported in part by the NSF.

References:

[GG] Adriano M. Garsia and Ira Gessel, *Permutation statistics and partitions*, Advances in Mathematics **31**(1979), 288-305.

[MZ] Philip Matchett Wood and Doron Zeilberger, *A Translation Method for Finding Combinatorial Bijections*, to appear in Annals of Combinatorics.
Available from `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/trans-method.html` .

[N] Moti Novick, *A Bijective proof of a major index theorem of Garsia and Gessel*, `http://arxiv.org/abs/0906.0377` .