The Maximal number of floors a Building can have where you can tell the highest floor from where you can throw a glass ball without breaking it, if you have b glass balls and are allowed t throws is $\binom{t}{b} + \binom{t}{b-1} + \ldots + \binom{t}{1}$

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The statement of the title is equivalent to:

Prop. Let $T_b(x)$ be the minimal number of throws needed to determine the cut-off floor, in an x-floor building, where it is no longer safe to throw down a glass ball without breaking it, if you have b glass balls that you are allowed to break. Then

$$T_b(x) = t \quad , \quad if \quad \binom{t-1}{b} + \binom{t-1}{b-1} + \ldots + \binom{t-1}{1} < x \leq \binom{t}{b} + \binom{t}{b-1} + \ldots + \binom{t}{1} \quad .$$

Proof: $T_b(x)$ obviously satisfies the **dynamical-programming** recurrence:

$$T_b(x) = \min_{1 \leq i \leq x} \quad max\,(\, T_{b-1}(i-1)\,,\, T_b(x-i)\,) \quad ,$$

but so does the function on the right hand side, as a routine but somewhat tedious verification shows. The proposition follows by induction on b and x, starting with the obvious initial conditions $T_1(x) = x$ and $T_b(0) = 0$. $\square$

Strategy: Without loss of generality let $x = \binom{t}{b} + \binom{t}{b-1} + \ldots + \binom{t}{1}$. The first ball has to be thrown from floor number $\binom{t-1}{b-1} + \binom{t-1}{b-2} + \ldots + \binom{t-1}{1} + 1$. If it breaks use the $b-1$ remaining balls and proceed recursively for the floors below it. If it does **not** break make this floor the ground floor, and use $t-1$ throws (and the b balls) to tackle the remaining $x = \binom{t-1}{b} + \binom{t-1}{b-1} + \ldots + \binom{t-1}{1}$ floors above it.

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