

**“TRIVIALIZING” GENERALIZATIONS OF SOME  
IZERGIN-KOREPIN-TYPE DETERMINANTS**

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ABSTRACT. We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson Condensation and/or Krattenthaler’s factor exhaustion method.

All our matrices will be assumed to be embedded inside an infinite matrix.

The first theorem adds *parameters* to the determinant formulas found in Kuperberg [Ku] (Theorem 15), as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laksov-Lascoux-Thorup, and Tsuchiya [T]. This way, the formulation is suited to the method of [AZ]. Our proofs are much more **succinct** and **automatable**, since their generality enables an easy induction using Dodgson’s rule [D, AZ], or by employing Krattenthaler’s elegant factor exhaustion method [Kr1]. Relevant background for this paper can found in [Ku], and references thereof.

**Theorem 1:**

$$\det \left( \frac{1}{x_i + y_j + Ax_i y_j} \right)_{i,j}^{1,n} = \frac{\prod_{i < j} (x_j - x_i)(y_j - y_i)}{\prod_{i,j}^n (x_i + y_j + Ax_i y_j)}.$$

$$\begin{aligned} & \det \left( \frac{1}{x_i + y_j} - \frac{1}{1 + x_i y_j} \right)_{i,j}^{1,n} \\ &= \frac{\prod_{i < j} (1 - x_i x_j)(1 - y_i y_j)(x_j - x_i)(y_j - y_i)}{\prod_{i,j}^n (x_i + y_j)(1 + x_i y_j)} \cdot \prod_{i=1}^n (1 - x_i)(1 - y_i). \end{aligned}$$

$$\begin{aligned} & \det \left( \frac{Ay_j + Bx_i}{y_j + x_i} \right)_{i,j}^{1,n} \\ &= (A - B)^{n-1} \frac{\left( A \prod_j^n y_j + (-1)^{n-1} B \prod_i^n x_i \right) \prod_{i < j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j}^n (x_i + y_j)}. \end{aligned}$$

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$$\begin{aligned} & \det \left( \frac{1 - x_i y_j}{y_j - x_i} \right)_{i,j}^{1,n} \\ &= (-1)^{\binom{n}{2}} \frac{\left( \prod_j^n (1 + x_j)(1 - y_j) + \prod_i^n (1 - x_i)(1 + y_i) \right) \prod_{i < j} (x_i - x_j)(y_i - y_j)}{2 \prod_{i,j}^n (y_j - x_i)}. \end{aligned}$$

**Sketch of Proof:** An automatic application of Dodgson Condensation [D, AZ].  $\square$

**Corollary (Cauchy, Stembridge, Laksov-Lascoux-Thorup)**

$$\begin{aligned} \det \left( \frac{1}{x_i + y_j} \right)_{i,j}^{1,n} &= \frac{\prod_{i < j} (x_j - x_i)(y_j - y_i)}{\prod_{i,j}^n (x_i + y_j)}. \\ Pf^2 &= \det \left( \frac{x_j - x_i}{x_j + x_i} \right)_{i,j}^{1,2n} = \prod_{i < j \leq 2n} \frac{(x_i - x_j)^2}{(x_i + x_j)^2}. \\ Pf^2 &= \det \left( \frac{x_j - x_i}{1 - x_i x_j} \right)_{i,j}^{1,2n} = \prod_{i < j \leq 2n} \frac{(x_i - x_j)^2}{(1 - x_i x_j)^2}. \end{aligned}$$

**Remark:** Notice that the latter two statements apply only to even-dimensional matrices. An error from [Ku] in the second formula has been corrected here.  $Pf$  stands for Pfaffian of a matrix.

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 [Ku] (typos corrected in [Kr2, Theorems 13 and 14]). Below,  $Z_1, Z_2, Z_3, Z_4, Z_5$  are defined by (here  $\gamma(a, b) = a - b, \tau(a, b) = a + b$ )

$$\begin{aligned} Z_1(p, q; x, y)_{i,j} &= \frac{\gamma(q^{j-i}, x^{j-i})}{\tau(p^{j-i}, y^{j-i})}, & Z_2(p, q; x, y)_{i,j} &= \frac{\gamma(q^{1+j-i}, x^{1+j-i})}{\tau(p^{1+j-i}, y^{1+j-i})}, \\ Z_3(p, q; x, y)_{i,j} &= \frac{\gamma(q^{-1+j-i}, x^{-1+j-i})}{\tau(p^{-1+j-i}, y^{-1+j-i})}, & Z_4(q, q; x, x)_{i,j} &= \frac{\gamma(q^{a+j-i}, x^{a+j-i})}{\tau(q^{a+j-i}, x^{a+j-i})}, \\ Z_5(q, q; x, x)_{i,j} &= \frac{\tau(q^{b+j-i}, x^{b+j-i})}{\gamma(q^{b+j-i}, x^{b+j-i})}, \end{aligned}$$

for  $a \in \mathbb{Z}, b = \pm n, \pm(n+1), \dots$

Let  $\delta_{e,n} = \frac{1+(-1)^n}{2}$  denote Kronecker's delta function centered at the even integers, and let  $\lambda_{i,j} = 1$ , if  $i \neq j$  and  $\lambda_{i,i} = 0$ .

**Theorem 2:** Write  $\gamma(a, b) = a - b, \tau(a, b) = a + b$ , we have the matrix determinants

$$\begin{aligned} \det \left( \frac{\gamma(q^{n+j-i}, x^{n+j-i})}{\gamma(p^{n+j-i}, y^{n+j-i})} \right)_{i,j}^{1,n} &= (py)^{\binom{n}{3}} \gamma(q, x)^n \frac{\prod_{j>i} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{\prod_{i,j} \gamma(p^{n+j-i}, y^{n+j-i})}. \\ \det \left( \frac{\tau(q^{j-i}, x^{j-i})}{\tau(p^{j-i}, y^{j-i})} \right)_{i,j}^{1,n} &= \frac{\prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \prod_{2 \nmid j-i>0} \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{(qx)^{\lfloor n^2/4 \rfloor} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}. \end{aligned}$$

$$\det(Z_1)_{i,j}^{1,n} = \delta_{e,n} \frac{\gamma(q, x)^n (py)^{n/2} \prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{(qx)^{\lfloor n^2/4 \rfloor} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.$$

$$\det(Z_2)_{i,j}^{1,n} = \frac{\gamma(q, x)^n \prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{(qx)^{\lfloor (n-1)^2/4 \rfloor} \tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.$$

$$\det(Z_3)_{i,j}^{1,n} = \frac{(-py)^n \gamma(q, x)^n \prod_{2|j-i>0} \gamma(p^{j-i}, y^{j-i})^2 \gamma(qp^{j-i}, xy^{j-i}) \gamma(xp^{j-i}, qy^{j-i})}{(qx)^{\lfloor (n+1)^2/4 \rfloor} \tau(p^n, y^n)^{1-\delta_{e,n}} \prod_{j>i} \tau(p^{j-i}, y^{j-i})^2}.$$

$$\det(Z_4)_{i,j}^{1,n} = 2^{n-1} \frac{q^{na} + (-1)^n x^{na}}{(qx)^{n(n-1)(n+1-3a)/6}} \frac{\prod_{j \neq i} \gamma(q^{|j-i|}, x^{|j-i|})}{\prod_{i,j} \tau(q^{a+j-i}, x^{a+j-i})}.$$

$$\det(Z_5)_{i,j}^{1,n} = (-1)^{\binom{n}{2}} 2^{n-1} \frac{q^{nb} + x^{nb}}{(qx)^{n(n-1)(n+1-3b)/6}} \frac{\prod_{j \neq i} \gamma(q^{|j-i|}, x^{|j-i|})}{\prod_{i,j} \gamma(q^{b+j-i}, x^{b+j-i})}.$$

$$Pf^2 = \det \left( \lambda_{i,j} \frac{\gamma(q^{j-i}, x^{j-i}) \gamma(r^{j-i}, z^{j-i})}{\gamma(p^{j-i}, y^{j-i})} \right)_{i,j}^{1,2n} = \frac{(yp)^n}{(qxrz)^{n^2}} \frac{\prod_{j \neq i}^{1,n} \gamma(p^{|j-i|}, y^{|j-i|})^2}{\prod_{i,j}^{1,n} \gamma(p^{n+j-i}, y^{n+j-i})^2} \times \prod_{i,j}^{1,n} \gamma(qp^{|j-i|}, xy^{|j-i|}) \gamma(xp^{|j-i|}, qy^{|j-i|}) \gamma(rp^{|j-i|}, zy^{|j-i|}) \gamma(zp^{|j-i|}, ry^{|j-i|}).$$

**Sketch of Proof:** Identities  $Z_4$  and  $Z_5$  are directly amenable to Dodgson’s Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader.  $\square$

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