In How Many Ways Can a King Walk n Steps?

Shalosh B. EKHAD [1]

Theorem. Let $f(n)$ be the number of ways a Chess King can walk n steps, returning to the starting point, in an "infinite" chessboard, then $f(0) = 1, f(1) = 0, f(2) = 8$ (obviously!) and for $n \geq 0$

$$f(n+3) =$$

$$32 \frac{(3n+7)(n+1)^2 f(n)}{(n+3)^2 (4+3n)} + 4 \frac{\left(27n^3 + 144n^2 + 248n + 139\right) f(n+1)}{(n+3)^2 (4+3n)} + \frac{(3n+5)(n+2)(3n+8) f(n+2)}{(n+3)^2 (4+3n)} .$$

We also have the asymptotic formula:

$$f(n) = \frac{2}{3\pi} \frac{8^n}{n} (1 - 4/9\, n^{-1} + 1/18\, n^{-2} + \frac{29}{486} n^{-3} + \frac{445}{17496} n^{-4} - \frac{443}{8748} n^{-5} - \frac{10933}{104976} n^{-6} +$$

$$\frac{35761}{944784} n^{-7} + \frac{3502795}{7558272} n^{-8} + \frac{13332763}{38263752} n^{-9} - \frac{30042779573}{11019960576} n^{-10}) + O(n^{-12}) .$$

Proof. Since *combinatorics is algebra*, and conversely *algebra is combinatorics*, $f(n)$ is the coefficient of $x^0 y^0$ in the polynomial $(x + 1/x + y + 1/y + xy + 1/(xy) + x/y + y/x)^n$, that can be expressed as a (formal!) double-contour integral. that is beautifully handled by Moa Apagodu and Doron Zeilberger's Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/MultiAlmkvistZeilberger` ,

that is explained in their article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/multiZ.pdf`.

Once we have the recurrence, the asymptotic was derived using the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/AsyRec`

that is briefly explained in Doron Zeilberger's article:

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/asy.pdf` .

Everything is rigorous *except* the constant in front, $\frac{2}{3\pi}$. It was found (empirically) by first estimating the constant in front (divide $f(n)$ by the expression on the right (except for the constant), and get a numerical estimate for that constant, then *identify* it using Maple's built-in command `identify`.)

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Of course, this can be easily rigorously obtained (either by hand, or automatically) by standard analytical (after (and possibly before) converting the double contour integral to a double trigonometric integral), or probabilistic methods (find the asymptotic covariance matrix, and use the local limit law) , but who cares? □

For the convenience of the readers, all the necessary tools have been combined into **one** self-contained Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/WalkPapers` ,

that can produce as many papers like this one as desired. Some other results can be gotten from the front of the present article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/melech.html` .

Comment. The first few terms of the sequence (staring with $n = 1$) are

$$0, 8, 24, 216, 1200, 8840, 58800, 423640, 3000480, 21824208, 158964960, 1171230984, 8668531872, 64574844048 \ldots$$

This is Sloane's `http://oeis.org/A098070`, where one can find a *fourth*-order, and hence, not as good as our *third*-order linear recurrence. As with many entries in Sloane's OEIS, one can't tell whether the recurrence is only *guessed* or actually *proved*. Of course, thanks to WZ theory, it is known *a priori*, in many cases, that a linear recurrence (with polynomial coefficients) *exists*, and hence we have a semi-rigorous meta-proof, that could be made completely rigorous that the guessed recurrence is provably correct, but it is (often) easier to use WZ (or multi-WZ) *ab initio*, because the later method gives you a recurrence **and its proof** at the *same time*.

A Quick Guide to the Maple package `WalkPapers`.

Once you have downloaded `WalkPapers` to your current directory, get into Maple, and type `read WalkPapers:` . For a list of the procedures, type `ezra();`, and for help with any specific procedure, type `ezra(ProcedureName)`.

The main procedure is `WalkPaper(S,n,m,K1,K2)`, where

(i) `S` is the set of steps (a set of lists of integers of the same size)

(ii) `n` is a symbol

(iii) `m` is a positive integer indicating the desired order for the asymptotic formula for $f(n)$.

(iv) $K1$ is a positive integer, indicating how many terms of the sequence you would like to have displayed.

(v) $K2$ is a large (recommended at least 1000) that is used to estimate, and identify the constant in front.

For example, the present article was gotten by typing:

`WalkPaper([1,0],[-1,0],[0,1],[0,-1],[1,-1],[-1,1],[1,1],[-1,-1],n,10,30,1000): .`

`WalkPaperSR(S,n,m,K1,K2)` is a semi-rigorous analog, that sometimes works faster.

`Sefer1D` and `Sefer2D` produce webbooks with lots of theorems for different sets of steps in one and two dimensions, respectively.