

THE QUANTUM MACMAHON MASTER THEOREM

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ABSTRACT. We state and prove a quantum-generalization of MacMahon's celebrated Master Theorem, and relate it to a quantum-generalization of the boson-fermion correspondence of Physics.

1. INTRODUCTION

1.1. MacMahon's Master Theorem. In this paper we state and prove a quantum-generalization of MacMahon's celebrated Master Theorem, conjectured by the first two authors and proved by the third.

Recall that in r -dimensional *quantum algebra* we have r indeterminates x_i ($1 \leq i \leq r$), satisfying the commutation relations $x_j x_i = q x_i x_j$ for all $1 \leq i < j \leq r$. We also consider matrices $A = (a_{ij})$ of r^2 indeterminates a_{ij} , $1 \leq i, j \leq r$, that commute with the x_i 's and such that for any 2 by 2 minor of (a_{ij}) , consisting of rows i and i' , and columns j and j' (where $1 \leq i < i' \leq r$, and $1 \leq j < j' \leq r$), writing $a := a_{ij}, b := a_{ij'}, c := a_{i'j}, d := a_{i'j'}$, we have the *commutation relations*:

$$(1) \quad ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \quad da = ad + (q - q^{-1})bc \quad .$$

We will call such matrices A *quantum matrices*.

Also recall that the *quantum determinant*, (first introduced in [FRT]) of any (not-necessarily quantum) r by r matrix $B = (b_{ij})$ may be defined by

$$\det_q(B) := \sum_{\pi \in S_r} (-q)^{-\text{inv}(\pi)} b_{\pi_1 1} b_{\pi_2 2} \cdots b_{\pi_r r},$$

where the sum ranges over the set of permutations, S_r , of $\{1, \dots, r\}$, and for any of its members, π , $\text{inv}(\pi)$ denotes the number of pairs $1 \leq i < j \leq r$ for which $\pi_i > \pi_j$.

Theorem 1. (Quantum MacMahon Master Theorem) *Fix a quantum matrix A of size r . For $1 \leq i \leq r$, let $X_i := \sum_{j=1}^r a_{ij} x_j$, and for any vector (m_1, \dots, m_r) of non-negative integers let $G(m_1, \dots, m_r)$ be the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}$ in $\prod_{i=1}^r X_i^{m_i}$. Then*

$$\sum_{m_1, m_2, \dots, m_r=0}^{\infty} G(m_1, \dots, m_r) = 1/\det_q(I - A).$$

The above result is not only interesting from the combinatorial point of view, but it is also a key ingredient in a *finite noncommutative formula* for the colored Jones function of a knot. This will be explained in a subsequent publication, [GL].

For an interpretation of the above theorem in terms of the so-called *boson-fermion correspondence*, see Section 3.

2. PROOF

The proof will make crucial use of a *calculus of difference operators*, developed by the third author in [Z]. Difference operators act on *discrete functions* J , that is functions whose domain is \mathbb{N}^r .

Before embarking on the proof, we need the following readily-verified facts.

Fact 1: For $1 \leq i < j \leq r$, $X_j X_i = q X_i X_j$.

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Fact 2: For each of the a_{ij} , define the operator Q_{ij} acting on expressions P involving a_{ij} by $Q_{ij}P(a_{ij}) := P(qa_{ij})$. Then, for any $1 \leq i, j \leq r$, and integer m_i and any expression F

$$x_i^{-m_i} X_j F = [(Q_{j1}^{-1} Q_{j2}^{-1} \cdots Q_{j,i-1}^{-1} Q_{j,i+1} \cdots Q_{jr})^{m_i} X_j] x_i^{-m_i} F.$$

Fact 3 (Column expansion with respect to the last column): Given an r by r quantum matrix (a_{ij}) , let A_i be the minor of the entry a_{ir} , i.e. the $r-1$ by $r-1$ matrix obtained by deleting the i^{th} row and r^{th} column. Then

$$\det_q(A) = \sum_{i=1}^r (-q)^{i-r} (\det_q A_i) a_{ir}.$$

Fact 4 (Equal columns imply that \det_q vanishes): In the notation of Fact 3, for all $j \neq r$,

$$\sum_{i=1}^r (-q)^{i-r} (\det_q A_i) a_{ij} = 0.$$

We are now ready for the proof. It is a quantum-adaptation of the ‘‘operator-elimination’’ proof MacMahon’s Master Theorem given in [Z].

Proof. (of Theorem 1) $G(m_1, \dots, m_r)$ is the coefficient of $x_1^0 \dots x_r^0$ in

$$H(m_1, \dots, m_r; x_1, \dots, x_r) := x_r^{-m_r} \cdots x_2^{-m_2} x_1^{-m_1} \prod_{i=1}^r X_i^{m_i}.$$

We will think of H as a *discrete function*, that is as a function of $(m_1, \dots, m_r) \in \mathbb{N}^r$. Let t_i ($i = 1, \dots, r$) be a new set of variables commuting with the x_i ’s and a_{ij} ’s. Substituting a_{ij} by $t_i a_{ij}$, it follows that the theorem is equivalent to the generating-function identity:

$$g(t_1, \dots, t_r) := \sum_{m_1, m_2, \dots, m_r=0}^{\infty} G(m_1, \dots, m_r) t_1^{m_1} \cdots t_r^{m_r} = 1 / \det_q(I - t_i a_{ij}).$$

Next we define the *shift-operators* M_i which act on a discrete function $J(m_1, \dots, m_r)$ by

$$(M_i J)(m_1, \dots, m_r) := J(m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_r).$$

Let’s see how M_i acts on H . By definition,

$$M_i H(m_1, \dots, m_r; x_1, \dots, x_r) = x_r^{-m_r} \cdots x_{i+1}^{-m_{i+1}} x_i^{-m_i-1} x_{i-1}^{-m_{i-1}} \cdots x_1^{-m_1} X_1^{m_1} \cdots X_{i-1}^{m_{i-1}} X_i^{m_i+1} X_{i+1}^{m_{i+1}} \cdots X_r^{m_r}.$$

By moving x_i^{-1} to the front and X_i in front of $X_1^{m_1}$, and using Fact 1 and $x_j x_i = q x_i x_j$, we have

$$M_i H(m_1, \dots, m_r) = q^{m_r + m_{r-1} + \cdots + m_{i+1} - m_1 - m_2 - \cdots - m_{i-1}} x_i^{-1} [x_r^{-m_r} \cdots x_1^{-m_1} X_i] X_1^{m_1} \cdots X_r^{m_r}.$$

By repeated use of Fact 2, this equals

$$q^{m_r + m_{r-1} + \cdots + m_{i+1} - m_1 - m_2 - \cdots - m_{i-1}} x_i^{-1} \cdot [(Q_{i2} \cdots Q_{ir})^{m_1} (Q_{i1}^{-1} Q_{i3} \cdots Q_{ir})^{m_2} (Q_{i1}^{-1} Q_{i2}^{-1} Q_{i4} \cdots Q_{ir})^{m_3} \cdots (Q_{i1}^{-1} Q_{i2}^{-1} \cdots Q_{i,r-1}^{-1})^{m_r} X_i] \cdot x_r^{-m_r} \cdots x_1^{-m_1} X_1^{m_1} \cdots X_r^{m_r},$$

which is equal to

$$q^{m_r + m_{r-1} + \cdots + m_{i+1} - m_1 - m_2 - \cdots - m_{i-1}} x_i^{-1} \cdot (q^{-m_2 - m_3 - \cdots - m_r} a_{i1} x_1 + q^{m_1 - m_3 - \cdots - m_r} a_{i2} x_2 + \cdots + q^{m_1 + m_2 + \cdots + m_{r-1}} a_{ir} x_r) H(m_1, \dots, m_r; x_1, \dots, x_r)$$

Multiplying out and rearranging, we get that the discrete function $H(m_1, \dots, m_r; x_1, \dots, x_r)$ is annihilated by the r operators ($i = 1, 2, \dots, r$)

$$\mathcal{P}_i := \sum_{j=1}^{i-1} -q^{-m_j - 2m_{j+1} - \cdots - 2m_{i-1} - m_i} a_{ij} x_j + (M_i - a_{ii}) x_i + \sum_{j=i+1}^r -q^{m_i + 2m_{i+1} + \cdots + 2m_{j-1} + m_j} a_{ij} x_j.$$

Now comes a nice surprise. Defining b_{ij} to be the coefficient of x_j in \mathcal{P}_i , it is immediately checked (using the fact that $M_i q^{m_i} = q(q^{m_i}) M_i$) that b_{ij} is also a quantum matrix! In other words it obeys the quantum commutation relations. Now we eliminate x_1, x_2, \dots, x_{r-1} by left-multiplying \mathcal{P}_i by the minor of b_{ir} in $B = (b_{ij})$ times $(-q)^{i-r}$, for each $i = 1, 2, \dots, r$, and adding them all up. By Fact 4, the coefficients of

x_1, \dots, x_{r-1} all vanish, and we are left with (after right-multiplying by x_r^{-1}) $\det_q(B)H(m_1, \dots, m_r) = 0$. Now using the definition of the quantum-determinant, it is readily seen that all the powers of q disappear! Hence

$$\det_q(M_i \delta_{ij} - a_{ij})H(m_1, \dots, m_r; x_1, \dots, x_r) = 0.$$

What's nice about the partial recurrence operator $\det_q(M_i \delta_{ij} - a_{ij})$ is that it does not depend on the x'_i 's, hence it also annihilates each and every single coefficient of H , in particular its constant term. Taking the constant term yields

$$\det_q(M_i \delta_{ij} - a_{ij})G(m_1, \dots, m_r) = 0.$$

But this is equivalent to the statement that the coefficient of a typical monomial $t_1^{\alpha_1} \dots t_r^{\alpha_r}$, with all the α_i strictly positive, in the formal power series

$$\det_q(I - t_i a_{ij})g(t_1, \dots, t_r),$$

vanishes. By induction (with respect to r), we already know that it is also true when some of the α_i are zero, *except* when they are all zero, in which case the coefficient of $t_1^0 \dots t_r^0$ is obviously 1. Hence

$$\det_q(I - t_i a_{ij})g(t_1, \dots, t_r) = 1.$$

□

3. BOSON-FERMION CORRESPONDENCE

Given a matrix $A = (a_{ij})$ of size r with commuting indeterminate entries which lie in a ring \mathcal{R} , the following identity

$$(2) \quad \frac{1}{\det(I - tA)} = \sum_{n=0}^{\infty} \text{tr } S^n(A)t^n$$

in $\mathcal{R}[[t]]$ in Combinatorics is called the *MacMahon Master Theorem Identity* (see [MM]) and in Physics it is called the *boson-fermion* correspondence. We may observe that

$$\det(I - tA) = \sum_{n=0}^{\infty} (-1)^n \text{tr } \Lambda^n(A)t^n$$

where $S^n(A)$ and $\Lambda^n(A)$ is the n th *symmetric* or *exterior power* of A , corresponding to bosons (that is, commuting particles) and fermions (that is, anti-commuting particles) respectively.

Now, we come to a q -analogue of the above. For a reference on quantum space and quantum algebra, see [Ka, Chapter IV] and [M].

Recall that a vector (column or row) of r indeterminate entries x_1, \dots, x_r lies in r -dimensional *quantum space* $A^{r|0}$ iff its entries satisfy

$$x_j x_i = q x_i x_j$$

for all $1 \leq i < j \leq r$.

Recall that an *endomorphism* of $A^{r|0}$ is a matrix $A = (a_{ij})$ of size r whose entries commute with the coordinates x_i of a vector $x = (x_1, \dots, x_r)^T \in A^{r|0}$ and in addition, Ax and $x^T A$ lie in $A^{r|0}$.

This is equivalent to the condition given before the statement of Theorem 1; see [Ka, Thm. IV.3.1]. The set of such matrices A are the points of the r -dimensional *quantum algebra* $M_q(r)$, which is defined to be the quotient of the free algebra in noncommuting variables x_{ij} for $1 \leq i, j \leq r$, modulo the left ideal generated by the commutation relations of Equation (1).

The algebra $M_q(r)$ has interesting and important structure. $M_q(r)$ is Noetherian, has no zero divisors, and in addition, a basis for the underlying vector space is given by the set of *sorted monomials* $\{\prod_{i,j} a_{ij}^{n_{ij}} \mid n_{ij} \geq 0\}$ where the product is taken lexicographically; see [Ka, Thm IV.4.1]. An important quotient of $M_q(r)$ is the *quantum group* $SL_q(r) := M_q(r)/(\det_q - 1)$, which is a Hopf algebra [Ka, Sec.IV.6] whose representation theory gives rise to the quantum group invariants of knots, such as the *Jones polynomial*.

Using the language of symmetric and exterior powers, Theorem 1 becomes the following:

Theorem 2. *If A is in $M_q(r)$, then*

$$\frac{1}{\det_q(I - tA)} = \sum_{n=0}^{\infty} \text{tr } S^n(A)t^n$$

Since the algebra $M_q(r)$ has a vector space basis given by sorted monomials, it should be possible to give an alternative proof of the quantum MacMahon Master Theorem using *combinatorics on words*, as was done in [FZ] for several proofs of the MacMahon Master theorem. We hope to return to this alternative point of view in the near future.

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