Proof of an Identity of Don Knuth

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Don Knuth, in a letter to Richard Stanley, Cc-d to several people, asked whether anybody has seen the following identity before:

$$\sum_{0\leq x_1,\ldots,x_n\leq 1} \frac{(z-x_1z^{n+1})(z^2-x_2z^{n+1})\ldots(z^n-x_nz^{n+1})z^{-nx_1-(n-1)x_2-\cdots-x_n}}{(x_1+1)(x_1x_2+x_2+1)\ldots(x_1\ldots x_n+\cdots+x_n+1)} = \frac{(1-z)(1-z^2)\ldots(1-z^{n+1})}{(n+1)!\,(1-z)^{n+1}}.$$

I have not. Here is a proof.

I will prove the more general identity:

$$A(n,r) := \sum_{0\leq x_1,\ldots,x_n\leq 1} \frac{(z-x_1z^{n+1})(z^2-x_2z^{n+1})\ldots(z^n-x_nz^{n+1})z^{-nx_1-(n-1)x_2-\cdots-x_n}}{(rx_1+1)(rx_1x_2+x_2+1)\ldots(rx_1\ldots x_n+x_2\ldots x_n+\cdots+x_n+1)} = \frac{(1-z)(1-z^2)\ldots(1-z^n)}{(1-z)^{n+1}}\cdot\sum_{l=1}^{n+1}\frac{z^{n-l+1}(1-z)^l r!}{(r+l-1)!(n-l+1)!}. \qquad (*)$$

The original identity follows by plugging $r=1$ and using the Binomial Theorem: $(z+(1-z))^{n+1}-z^{n+1}=1-z^{n+1}$.

To prove the identity for $A(n,r)$, split the sum according to whether $x_1=0$ or $x_1=1$ to get the recurrence:

$$A(n,r) = z^n A(n-1,1) + \frac{(1-z^n)}{r+1}A(n-1,r+1) \quad .$$

Since $A(0,r)=1$, and the expression on the right of $(*)$ obviously also satisfies this recurrence, we are done. □

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