

# Lagrange Inversion Without Tears (Analysis) (based on Henrici)

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**Def 1:** A formal Laurent series (f.L.s) is

$$\sum_N^{\infty} a_n x^n ,$$

where  $a_n$  and  $x$  are symbols and  $N$  is an integer.

**Def 2:**  $[x^n]f(x)$  is the coeff. of  $x^n$  in  $f(x)$ .

**Def 3:**  $Res f(x)$  is the coeff. of  $x^{-1}$  in  $f(x)$ .

**Def 4:** Given a sequence  $f_i$  of f.L.s. starting at  $N_i$ , their sum  $\sum f_i$  makes sense if the  $N_i$  are bounded below and for every  $n$ , the set  $\{[x^n]f_i(x)\}$  has only finitely many non-zero terms, and then the coeff. of  $x^n$  in sum  $\sum f_i$  is by definition, that sum.

**Def. 5**

$$cx^N \sum_M^{\infty} a_n x^n := \sum_M^{\infty} ca_n x^{n+N} :$$

(convince yourself that that the rhs makes sense)

**Def. 6** If  $f(x) = \sum_N^{\infty} a_n x^n$  is a f.L.s., and so is  $g(x)$  then

$$f(x)g(x) := \sum_N^{\infty} a_n x^n g(x)$$

(convince yourself that that the rhs makes sense).

**Def. 7**

$$\left(\sum_N^{\infty} a_n x^n\right)' := \sum_N^{\infty} na_n x^{n-1}$$

**Prop. 1** (Product Rule):  $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

**Proof:** True for  $f(x) = x^m, g(x) = x^n$ , and hence in general, since both sides are linear in  $f(x)$ , and in  $g(x)$ .

**Prop. 2:**  $(f(x)^k)' = kf(x)^{k-1}f'(x)$

**Proof:** True for  $k = 1$ , and then by induction on  $k$ , for positive  $k$  and for negative  $k$  by using the product rule applied to  $1 = f(x)^k f(x)^{-k}$ .

**Prop. 3:** (Chain Rule): If  $\Phi$  and  $f(x)$  are f.L.s. then  $(\Phi(f(x)))' = \Phi'(f(x))f'(x)$ .

**Proof:** True for  $\Phi = x^k$  thanks to Prop.2, now extend by linearity.

**Prop. 4:** If  $f(x)$  is a f.L.s. then  $Res(f'(x)) = 0$ .

**Proof:**  $(x^n)' = nx^{n-1}$  can never be a multiple of  $x^{-1}$ , hence true for monomials, and by linearity for all  $f(x)$ .

**Prop. 5** (Integration by parts) If  $f(x)$  and  $g(x)$  are f.L.s. then  $Res(f'(x)g(x)) = -Res(f(x)g'(x))$ .

**Proof:** By Prop. 1 and 4.

**Prop. 6** (change of variables): Let  $u(t)$  be a f.L.s starting at  $t$  (i.e.  $N = 1$ ) and  $\Psi(z)$  be any f.L.s. then  $Res_t(u'(t)\Psi(u(t))) = Res_z\Psi(z)$

**Proof:** By linearity enough to prove it for monomials  $\Psi(z) = z^k$ . Both sides are 0 if  $k \neq -1$ , the right by definition 3, the left by Prop. 4. When  $k = -1$  the right is 1, by definition and the left is  $Res(u'(t)/u(t)) = (u_1 + 2u_2t + \dots)/(u_1t + u_2t^2 + \dots) = 1/t + O(1)$ .

**Theorem (Lagrange Inversion Theorem):** If  $u(t)$  and  $\Phi(t)$  are f.L.s. starting at  $t$  and  $t^0$  respectively, then  $u(t) = t\Phi(u(t))$  implies  $[t^n]u(t) = (1/n)[z^{n-1}]\Phi(z)^n$

**Proof:**

$$\begin{aligned}
 [t^n]u(t) &= Res_t(u(t)t^{-n-1}) = Res_t(u(t)(t^{-n}/(-n))') \stackrel{(5)}{=} (1/n)Res_t(u(t))'t^{-n} \stackrel{given}{=} \\
 &(1/n)Res_t(u'(t)(\Phi(u(t))/u(t))^n) \stackrel{(6)}{=} (1/n)Res_z(\Phi(z)^n/z^n) = (1/n)[z^{n-1}]\Phi(z)^n \quad \square.
 \end{aligned}$$