A SHORT PROOF OF MCDOUGALL’S CIRCLE THEOREM

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In fond memory of Andrei Zelevinsky (1953-2013) who loved Ptolemy’s Theorem.

Ptolemy’s Theorem is a beautiful, classical result concerning quadrilaterals. Specifically, let \( P_1, P_2, P_3, P_4 \) be cyclically ordered points on a circle and \( d_{i,j} \) denote the distance between \( P_i \) and \( P_j \). Then Ptolemy’s Theorem states
\[
d_{1,3}d_{2,4} = d_{1,2}d_{3,4} + d_{1,4}d_{2,3}.
\]
Refinements known as the Brahmagupta-Mahavira identities [1] lead to a “ratio version” of Ptolemy’s Theorem:
\[
\frac{d_{1,3}}{d_{2,4}} = \frac{d_{1,2}d_{1,4} + d_{2,3}d_{3,4}}{d_{1,4}d_{3,4} + d_{1,2}d_{2,3}}.
\]
Equation (1) can be written as
\[
\frac{1}{d_{1,2}d_{1,4}} + \frac{1}{d_{1,2}d_{2,3}d_{2,4}} = \frac{1}{d_{1,3}d_{2,3}d_{3,4}} + \frac{1}{d_{1,4}d_{2,4}d_{3,4}}.
\]
Jane McDougall [2] has generalized this result from 4 points to 2n points.

**Theorem 1.** (McDougall) Let \( n \) be a positive integer and \( P_i, 1 \leq i \leq 2n \), be cyclically ordered points on a circle. Furthermore, let
\[
R_i := \prod_{1 \leq j \leq 2n, j \neq i} d_{i,j}.
\]
Then
\[
\sum_{i=1}^{n} \frac{1}{R_{2i}} = \sum_{i=1}^{n} \frac{1}{R_{2i-1}}.
\]
McDougall’s proof, using tools from complex analysis, follows by applying harmonic mappings to a class of minimal surfaces. Related use of these methods can be found in [3]. The goal of this note is to provide a short, elementary proof of McDougall’s theorem. The key to the proof is using the Lagrange Interpolation Formula: If \( P(z) \) is a polynomial whose degree does not exceed \( N - 1 \), then
\[
P(z) = \sum_{i=1}^{N} \frac{(z - z_1)\cdots(z - z_{i-1})(z - z_{i+1})\cdots(z - z_N)}{(z_i - z_1)\cdots(z_i - z_{i-1})(z_i - z_{i+1})\cdots(z_i - z_N)} P(z_i)
\]
for any distinct numbers \( z_1, \ldots, z_N \). Using a formula like this is not surprising since Equation (2) involves sums of products.
Proof. Without loss of generality, assume the circle is centered at the origin and has radius one. Denote the points as \( P_i = (\cos 2t_i, \sin 2t_i) \), \( 1 \leq i \leq 2n \), where \( 0 \leq t_1 < t_2 < \cdots < t_{2n} < \pi \). Basic trigonometry can be used to show that \( d_{i,j} = 2\sin(t_i - t_j) \). Letting \( u_i = e^{it_i} \) where \( I = \sqrt{-1} \), it follows that
\[
d_{i,j} = -I \left( \frac{u_i^2 - u_j^2}{u_i u_j} \right)
\]
whenever \( i < j \). This produces
\[
I^{2n-1}(-1)^i \frac{1}{R_i} = \left( \prod_{j=1}^{2n} u_j \right) \frac{u_i^{2n-2}}{\prod_{j \neq i} (u_i^2 - u_j^2)}.
\]

With Equation (2) in view, we construct
\[
I^{2n-1} \sum_{i=1}^{2n} (-1)^i \frac{1}{R_i} = \left( \prod_{j=1}^{2n} u_j \right) \sum_{i=1}^{2n} \frac{u_i^{2n-2}}{\prod_{j \neq i} (u_i^2 - u_j^2)}.
\]

Applying the Lagrange Interpolation Formula with \( P(z) = z^r \), where \( r < N - 1 \), the coefficient of \( z^{N-1} \) yields
\[
0 = \sum_{i=1}^{N} \left( z_i - z_1 \right) \cdots \left( z_i - z_{i-1} \right) \left( z_i - z_{i+1} \right) \cdots \left( z_i - z_N \right).
\]

Taking \( N = 2n \), \( r = n - 1 \) and \( z_i = u_i^2 \) for \( i = 1, 2, \ldots, 2n \), the \( z_i \) terms are distinct and the right side of Equation (3) must equal zero. This is equivalent to the desired result. \( \square \)

Remark 1. The Lagrange Interpolation Formula is easy to prove, so no sophisticated machinery is needed for proving McDougall’s Theorem.

Remark 2. Letting the circle’s radius approach infinity gives the corollary that Equation (2) holds if the \( 2n \) points are collinear. In fact, the equation still holds on lines even if the number of points is odd:
\[
\sum_{i=1}^{2n-1} (-1)^i \frac{1}{R_i} = 0.
\]

We leave this as an exercise for the reader.

References

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