NSF Proposal: Symbolic Computation and Combinatorics

NSF PROPOSAL: SYMBOLIC COMPUTATION and COMBINATORICS

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Project Summary

SYMBOLIC COMPUTATION and COMBINATORICS

Doron Zeilberger proposes to continue to develop methodologies for harnessing the great potential of Symbolic Computation to do research in Combinatorics and related areas. In particular he hopes to introduce new computational and conceptual frameworks that would extend the so-called Wilf-Zeilberger proof theory to much wider classes of identities and theorems. He also proposes to continue his efforts in ‘Artificial Combinatorics’, and develop algorithms for the discovery and rigorous proof of theorems in combinatorics whose complexity make them unfeasible for human proofs. This research should be symbiotic, as it is expected that both the concrete results and the underlying methodologies, would help computer algebra developers to improve and enhance their systems.

Note: Previous grants of Zeilberger were supported by the Algebra and Number Theory program, with split-funding from Computational Mathematics and Numeric and Symbolic Computations (Computer Science). The present proposal may also be considered for such split-funding.
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Summary of Results from Previous NSF Support:DMS-9500646 and DMS-9732602

1. The current NSF award number is DMS-9732602 for the period 1997-2000, totaling $180,000.

2. Its title was: “Targeted Proof Machines in Combinatorics”.

3. Summary of the results of the completed work.

(The numbered references apply to the list of papers written with the NSF support of the above grants, given at section 4. The lettered references are to papers given at the end of the section “Proposed Research”.)

Symbolic Computation played a larger and larger role in my research efforts. First, several interesting extensions of WZ theory (that was recognized by the award of the 1998 Steele prize for research to Herb Wilf and myself) were made, mostly in collaboration with my students Tewodros Amdeberhan and Akalu Tefera, both of whom were supported at various times by this and the previous NSF grant. Tewodros Amdeberhan and I investigated applications of WZ theory to irrationality proofs, convergence-acceleration formulas, and determinant evaluation. Akalu Tefera developed a very powerful Maple package \texttt{MultiInt} that implements the continuous WZ theory for an arbitrary number of variables. Using this package, Tefera found a very interesting new Selberg-like integral formula for several variables.

Computer Algebra was also crucial in the design of the very complicated proof of the Alternating Sign Matrix Conjecture([?]), and in that of the proof of the Refined Alternating Sign Matrix Conjecture([4]), where the punch-line was supplied by the Maple package EKHAD. Both of these proofs are beautifully described (along with background material and Kuperberg’s beautiful simpler proof of the (unrefined, original) ASM conjecture) in Dave Bressoud’s recent book [Br], that won the MAA Bekenbach Book Prize. A nice shorter account was written by Bressoud and Jim Propp ([BP]).

My student Aaron Robertson, who was also supported by this grant for one year, and I, proved a conjecture of Ron Graham about Schur triples ([20]), thereby sharing (with T. Schoen) the prize of $100 that Graham offered. Once again, computer algebra (the Maple package \texttt{ROH}) was very important for designing the proof.

My computer, Shalosh B. Ekhad, and I, improved the lower bound for the asymptotics of the number of ternary square-free words ([18]). Together we also proved a conjecture of Alexander Kirillov ([8]). Using WZ theory, Shalosh([22]) proved a conjecture of Scott Ahlgren and Ken Ono, that was needed to finish up a deep conjecture of Frits Beukers ([AO]).

With Dominique Foata, who served as a consultant to this grant, we published several papers in pure combinatorics ([5][10][29][35]). We are especially proud of [29], where a generalization of Hyman Bass’s theorem is proved that was subsequently used by Lin and Wang ([LW]) in Knot Theory.
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Shalosh B. Ekhad and I used clever programming and about two weeks of computer time to prove the long-lost ‘Cosmological Theorem’, about the celebrated 1,11,21,1211,111221, ... sequence, supposedly proved, but lost, by Conway ([13]). Because of the length and complexity of the computer proof, it is unlikely that the lost (human) ‘proof’ was complete. Also of the same flavor is the automation of the three-rowed case of David Gale’s game of Chomp ([34]).

Other attempts at ‘artificial combinatorics’ are [17], that automatically finds enumeration schemes for enumerating so-called Wilf classes, and [26] that automatically counts Lego towers, or more prosaically, generalized vertically convex polyominoes. In [27] and [33] the very powerful Goulden-Jackson method is extended and implemented.

In [28], a ‘futuristic’ Geometry textbook is given. It is now incorporated in the much larger, and continuously evolving Maple package RENE, available from the proposer’s website.

In [24], a conjecture of Robbins and Chan was proved. Alex Postnikov and Richard Stanley ([PS]) found interesting ramifications in algebraic combinatorics.

In [30] we give algorithms for finding generating functions for enumerating lattice animals, and self-avoiding polygons and walks of restricted width (both globally and locally).

But the work that I believe to be the most significant, and whose further development constitutes a large part of my planned research to be described below, is the Umbral Transfer-Matrix method ([32]). It combines Gian-Carlo Rota’s seminal notion of umbra with the Transfer-Matrix method, to produce functional equations, as opposed to mere algebraic equations.
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4. List of Publications resulting from the NSF awards in 1996-2000

1996


8. (With S. B. Ekhad) An explicit formula for the number of solutions of \( X^2 = 0 \) in triangular matrices over \( GF(q) \), Elec. J. Comb. 3 R2.

1997


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J. of Combinatorics 4(2), [Wilf Festschrift volume], R22.


1998


1999


28. (With S. B. Ekhard) PLANE GEOMETRY: An Elementary School Textbook (ca. 2050), Mathematical Intelligencer 21(3), 64-70.

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2000

30. Symbol-Crunching with the Transfer-Matrix Method in Order to Count Skinny Physical Creatures, INTEGERS, 0, A9.


35. Babson and Steingrímsson’s permutation statistics are indeed Mahonian, (and sometimes even Euler-Mahonian), submitted.

5. Many of my papers are accompanied by Maple packages that are available, free of charge, from my homepage http://www.math.temple.edu/~zeilberg/. In addition, there are quite a few packages that belong to forthcoming papers, or stand by themselves. Some of them are of a rather general scope, and should be useful to researchers in combinatorics, number theory, analysis, statistical physics, and possibly other areas.

6. A large part of the proposed research is a direct continuation of the previous research, but there are also new directions, in which the connection is less obvious.

7. Education and Human Resources Statement.

In 1997, my Ph.D. students John Majewicz, Tewodros Amdeberhan, and John Noonan finished. Amdeberhan and Noonan were each supported for one year in the penultimate grant. John Majewicz has accepted an Associate Professorship at the Community College of Philadelphia. John Noonan is Assistant Professor at Mount Vernon Nazarene College, Ohio, and Tewodros Amdeberhan declined an offer for a postdoc at Penn-State in favor of a tenure-track appointment at DeVry Inst. of Tech., NJ. He is currently associate professor there.

All these theses combined experimental mathematics with more theoretical investigations and used computer algebra heavily.

Last year (spring 1999), Aaron Robertson and Melkamu Zeleke, who were each supported for one year by this grant, graduated. Aaron accepted a tenure-track appointment at Colgate University. I am very pleased that he is continuing to do very deep research, both computational and theoretical, in Ramsey theory. Melkamu is currently tenure-track assistant professor at William Patterson
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University. Melkamu is collaborating very nicely with colleagues in bijective combinatorics.

This year (spring 2000) Akalu Tefera and Anne Edlin received their Ph.D. under my direction. Akalu just started a tenure-track assistant professorship at Grand Valley State University (in Michigan), and Anne is assistant professor at Holy Family College (New Jersey). As I mentioned above Akalu developed sophisticated software for the continuous WZ method. Anne extended and implemented the Goulden-Jackson method to cyclic words.

Right now I only have one student, Mohamud Mohammed, who is very strong and promising. I also have several other strong prospective students. The research conducted by my students involves large-scale computing that should yield software of wide interest and applicability to the mathematical community. I am hence requesting student support, on the level of one student, that would be split between my students, and would free them from some teaching obligations.

I am the local expert on computer algebra. Since 1988, I have been teaching both graduate and undergraduate courses that were very well attended, in using Maple and Mathematica to do research in mathematics. Since most of the graduate students that attend my classes are also teaching assistants, this know-how gets transmitted to the undergraduates.

Paper [2] above was studied in a workshop for gifted high-school students conducted at MIT by Satomi Okazaki, and one of the students, Lauren Williams (currently a Sophomore at Harvard), used its method to write a paper (that later appeared in the Elec. J. Combinatorics), that won third prize in the 1996 International Science Fair. Lauren is currently a senior at Harvard and plans to specialize in combinatorics.
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PROPOSED RESEARCH: SYMBOLIC COMPUTATION and COMBINATORICS

Introduction

While the so-called Wilf-Zeilberger (WZ) proof theory and the holonomic paradigm, that has been successfully implemented in all the major computer algebra systems, and are widely used by mathematicians and scientists, can handle a rather wide class of summations and integrations, I feel that it is still but the tip of an iceberg. In this proposal, I will describe my plans to extend WZ theory to sums and integrals with arbitrarily many summation and integration signs, like the celebrated Selberg and Mehta integrals and the Macdonald-Dyson constant term identities for the infinite families of root systems. I also hope to go far beyond the holonomic paradigm, trying to find wider and wider computationally decidable ansatzes.

I also hope to continue to do ‘Artificial Combinatorics’. Here one looks at a class of combinatorial problems, for example that of enumerating so-called Wilf classes, and tries to teach the computer how to do research, by formalizing methods and tricks used, usually implicitly, in human research, and programming them. Once the computer ‘learned’ how to perform these ‘tricks’ it can go much further, of course.

In addition, I also hope to continue my work on a general theory of permutation statistics, in collaboration with Dominique Foata.

WZ Theory: Chapter II

Recall the ([WZ])

Fundamental Theorem of Multi-WZ Theory (Discrete Version)

‘Whenever’ we want to prove an identity of the form:

\[ \sum_{k_1} \ldots \sum_{k_r} NICE(n; k_1, \ldots, k_r) \equiv 1 \ , \]

There exist \( NICE_1, NICE_2, \ldots, NICE_r \) such that

\[ \Delta_n NICE = \sum_{i=1}^{r} \Delta_{k_i} NICE_i \]

Furthermore, \( R_i := NICE_i / NICE \), \( (i = 1, \ldots, r) \), are rational functions of \( (n; k_1, \ldots, k_r) \).

We also have the :

Fundamental Theorem of Multi-WZ Theory (Continuous Version)

‘Whenever’ we want to prove an identity of the form:

\[ \int \ldots \int NICE(n; x_1, \ldots, x_r) \ dx_1 \ldots dx_r \equiv 1 \ , \]
there exist $NICE_i^j(n; x_1, \ldots, x_r)$ ($i = 1, \ldots, r$), such that

$$\Delta_n NICE = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} NICE_i^j .$$

Furthermore, $R_i := NICE_i^j/NICE$ are rational functions of $(n; x_1, \ldots, x_r)$.

Note that the mere existence of $NICE_i^j$ (or equivalently the $R_i$), together with the trivially verifiable case $n = 0$, implies the identity, since,

$$\Delta_n \int \cdots \int NICE = \int \cdots \int \Delta_n NICE = \sum_{i=1}^{r} \int \cdots \int \frac{\partial}{\partial x_i} \{NICE_i^j\} = 0 ,$$

(because the $NICE_i^j$ are of compact support).

Two famous examples are

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{r} \frac{t_i^a}{(1 - t_i)^y} \prod_{1 \leq i < j \leq r} (t_i - t_j)^{2z} dt_1 \cdots dt_r
= \prod_{j=1}^{r} \frac{(x + (j - 1)z)! (y + (j - 1)z)! (jz)!}{(x + y + (r + j - 2)z + 1)!} . \quad (Selberg)$$

and

$$\int_{|z_1| = 1} \cdots \int_{|z_r| = 1} \left\{ \prod_{1 \leq i \neq j \leq r} \frac{(1 - z_i/z_j)^a}{z_i} \right\} \frac{dz_1}{z_1} \cdots \frac{dz_r}{z_r} = (ra)! . \quad (Dyson)$$

[This identity has a good pedigree. It was conjectured in 1960 by Dyson, and proved a year later by Gunson, and by Ken Wilson of Nobel (Renormalization Group) fame. The ‘book’ proof was given, in 1970, by the great statistician I.J. Good.]
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of hypergeometric function (what I called ‘nice’) of \( r \) variables, where \( r \) is not merely a symbol denoting an integer, but is a symbol period. Prototype nice functions should be:

\[
\left( \sum_{i=1}^{r} n_i \right)!
\]

\[
\prod_{i=1}^{r} NICE(n_i)
\]

and, probably

\[
\prod_{i=1}^{r-1} NICE(n_i + n_{i-1}),
\]

\[
\prod_{1 \leq i < j \leq r} NICE(n_i + n_j),
\]

where \( NICE \) is a nice function of a single discrete variable.

The ‘language’ should contain, as primitive symbols, \( \prod \) and \( \sum \), that would have to be incorporated into the algorithm.

We also need a notion of ‘global’ niceness for \( NICE_1, \ldots, NICE_r \), i.e., it does not suffice that \( NICE_i \) would be nice in its arguments, but we should insist that \( NICE_i \), when also viewed as a function of its subscript \( i \), is nice, in a sense yet to be made precise. Somehow, we should also bring in symmetry.

So hopefully, if this plan succeeds, Macdonald’s constant term identities, the Mehta integral, the Selberg integral, and their kin, would be fully automated.

Beyond the Holonomic Paradigm

Recall that a continuous function of one variable \( f(x) \) is holonomic (a.k.a. D-finite) if it satisfies a homogeneous linear differential equation with polynomial coefficients. For functions of several variables one insists on equations on each variable. While this class includes all polynomials and functions of hypergeometric type, and hence most of the classical functions of mathematical physics, it is still a rather special class.

It should be interesting to explore larger classes that would also carry their own proof theory. One obvious choice would be solutions of differential equations with holonomic coefficients, that one could call super holonomic. The simplest such function is \( e^{x} \). However, preliminary investigations show that a richer theory can be obtained by considering the narrower class of solutions of differential equations with algebraic coefficients.

Another interesting problem would be to extend WZ theory to multibasic sums, i.e. \( q \)-hypergeometric sums with several bases.
Enumeration Schemes

Suppose that we have to find a ‘formula’ (in the sense of Wilf, i.e. a polynomial-time algorithm) for computing \( a_n := |A_n| \), where \( A_n \) is an infinite family of finite sets, parameterized by \( n \). Usually \( A_n \) is a natural subset of a larger set \( B_n \), and is defined as the set of members of \( B_n \) that satisfy a certain set of conditions \( C_n \). For example if \( A_n \) is the set of permutations on \( \{1, 2, \ldots, n\} \), then \( B_n \) may be taken as the set of words of length \( n \) over the alphabet \( \{1, 2, \ldots, n\} \), and \( C_n \) can be taken as the condition: ‘no letter can appear twice’. A naive algorithm for enumerating \( A_n \) would be to actually construct the set, by examining the members of \( B_n \), one by one, checking whether they satisfy \( C_n \), and admitting those that qualify. Then \( a_n = \text{Cardinality of } A_n \).

But a much better approach would be to find a structure theorem that expresses \( A_n \), using unions, Cartesian products, and possibly complements, of well known sets. Failing this, it would be also nice to express \( A_n \), recursively, in terms of \( A_{n-1}, A_{n-2}, \ldots \), and easy-to-count sets, getting a recurrence formula. Going back to the permutation example, Levi Ben Gerson proved the structure theorem \( A_n = \{1, 2, \ldots, n\} \times A_{n-1} \), from which he deduced the recurrence \( a_n = na_{n-1} \), enabling a polynomial-(in fact linear-) time algorithm for computing \( a_i \), for \( 1 \leq i \leq n \).

Alas, this is not always easy, and for many enumeration sequences, e.g. the number of self-avoiding walks, may well be impossible, and who knows, perhaps one day even provably impossible.

It is conceivable, however, that a combinatorial family \( A(n) \), does not possess a recursive structure by itself, but by refining it, using a suitable parameter, one can partition \( A(n) \) into the disjoint union:

\[
A(n) = \bigcup_{i=1}^{n} B(n, i),
\]

and try and find a structure theorem for the two-parameter family \( B(n, i) \). This will imply a recurrence for the cardinalities \( b(n, i) := |B(n, i)| \), that would enable a fast algorithm for \( a(n) = \sum_{i=1}^{n} b(n, i) \).

Sometimes, not even the \( B(n, i) \) suffice. Then we could try to partition \( B(n, i) \) into the following disjoint union:

\[
B(n, i) = \bigcup_{j=1}^{i-1} C_1(n, i, j) \cup \bigcup_{j=i+1}^{n} C_2(n, i, j),
\]

and try to express \( C_1(n, i, j) \) and \( C_2(n, i, j) \) in terms of \( A(m), B(m, i'), C_1(m, i', j'), \) and \( C_2(m, i', j') \), with \( m < n \). One can keep going indefinitely. If this process halts after a finite number of refinements, then we have indeed a formula (in the sense of Wilf) for \( a(n) \).

In [17] I showed how to enumerate Wilf classes, but the ‘success rate’ was only about 20 percents. I propose to refine and enhance the notion of enumeration schemes to make the success rate closer to 100 percents. I also hope to use the same circle of ideas in enumerating words, special kinds of graphs, and other combinatorial families.
The Umbral Transfer-Matrix Method

The Finite Transfer-Matrix method is used to weight-enumerate paths on a finite digraph, see [30] for a detailed exposition. Here, we will be considering directed graphs whose set of vertices, \( V \), is infinite, with possibly multiple edges. Even though there are infinitely many vertices, we will assume that they can be partitioned into a finite union of vertex-families, \( \{v_1, \ldots, v_n\} \), such that each family \( v_i \) is an \( l_i \)-parameter infinite family, parameterized by the \( l_i \) discrete variables \((a_1, \ldots, a_{l_i})\), where \((a_1, \ldots, a_{l_i})\) ranges over a well-defined subset \( D_i \) of \( \{0,1,2,3,\ldots\}^{l_i} \). So the vertex set of our infinite digraph can be partitioned as follows

\[
V = \bigcup_{i=1}^{n} \bigcup_{(a_1, \ldots, a_{l_i}) \in D_i} v_i(a_1, \ldots, a_{l_i}) .
\]

We will also assume that for any pair of vertex types \( v_i \) and \( v_j \), there are \( K(i,j) \geq 0 \) families of edges, and for each of \( k = 1, \ldots, K(i,j) \), the type-\( k \) edge coming out of vertex \( v_i(a_1, \ldots, a_{l_i}) \) may wind up in any of the vertices \( v_j(b_1, \ldots, b_{l_j}) \), where \((b_1, \ldots, b_{l_j})\) may belong to a well-defined subset of \( D_j \), let’s call it \( E^{(k)}_{i,j} (a_1, \ldots, a_{l_i}) \). We also assume that every such edge has a certain weight given by a weight-function

\[
W_{i,j}^{(k)}(a_1, \ldots, a_{l_i}; b_1, \ldots, b_{l_j}) .
\]

The weight of a path \( P \), \( Wt(P) \), is the sum of the weights of its participating edges. We are interested in computing the weight-enumerator of all paths

\[
\sum_{P \in \text{Paths}} q^{Wt(P)},
\]

either explicitly, and failing this, to have a polynomial-time algorithm for computing the series expansion, i.e. the first \( N \) terms of its power-series expansion, for any given \( N \).

We now digress to define a Rota-Operator.

Definition of an Atomic Rota-Operator: An Atomic Rota operator from the ring of formal power series in \( r \) variables \( Z(q)(x_1, \ldots, x_r) \) to the ring of formal power series in \( s \) variables \( Z(q)(y_1, \ldots, y_s) \) (with coefficients from the ring of integer-coefficient formal power-series in \( q \)), is an operator of the form

\[
T[f(x_1, \ldots, x_r)] = R(q,y_1, \ldots, y_s)D^{\alpha_1}_{x_1} \ldots D^{\alpha_r}_{x_r} f(x_1, \ldots, x_r) |_{\{x_1 = m_1, \ldots, x_r = m_r\}}, \quad (\text{ARO})
\]

where \( R(x,q_1, \ldots, x_r) \) is a rational function of all its arguments, \( D_{x_1}, \ldots, D_{x_r} \) are the differentiation operators with respect to \( x_1, \ldots, x_r \) respectively, \( \alpha_1, \ldots, \alpha_r \) are non-negative integers, and \( m_1, \ldots, m_r \) are each monomials in the variables \((q,y_1, \ldots, y_s)\).

An Example of an Atomic Rota Operator:

\[
f(x_1, x_2) \rightarrow \frac{q^3 y_1 y_2 y_3}{(1 - q y_1)(1 - q^2 y_1 y_2 y_3)} D_{x_1} D_{x_2} f(y_1 y_2 y_3, q y_3) .
\]
Definition of a Rota Operator: A Rota Operator is a sum of Atomic Rota Operators.

It turns out that in many applications, the following property holds:

The Umbral Axiom

For every pair of vertex types, \( v_i, v_j \), and for each of its \( K(i, j) \) edge-types connecting them, the following operator from \( Z(q)(x_1, \ldots, x_i) \) to \( Z(q)(y_1, \ldots, y_j) \), defined on the basis of monomials by

\[
x_1^{a_1} \cdots x_i^{a_i} \rightarrow \sum_{(b_1, \ldots, b_j) \in E^k_{i,j}(a_1, \ldots, a_i)} q^{W_{i,j}^k(a_1, \ldots, a_i, b_1, \ldots, b_j)} y_1^{b_1} \cdots y_j^{b_j}
\]

is an atomic Rota operator, let’s call it \( Q_{i,j}^k \).

Also, let’s define the transition-operator from vertices of type \( i \) to vertices of type \( j \) (\( 1 \leq i, j \leq n \)), by

\[
\mathcal{P}_{i,j} := \sum_{k \in K(i,j)} Q_{i,j}^k
\]

which by our assumption are all Rota operators.

Let’s define the mishkal of a path \( P \), in our digraph, that ends with the vertex \( v_i(a_1, \ldots, a_i) \) by

\[
q^{W_i(P)} x_1^{a_1} \cdots x_i^{a_i},
\]

and let’s define the total mishkal of all the paths that end in a type-\( i \) vertex by

\[
F_i(q; x_1, \ldots, x_i) := \sum_P \text{mishkal}(P),
\]

where the sum extends over the infinite set of paths that end in a type-\( i \) vertex.

It follows immediately from this set-up that the \( n \) formal power series \( F_j \), (\( j = 1, \ldots, n \)) satisfy the following system of \( n \) differential-functional equations

\[
F_j = [j \in \text{Start}] + \sum_{i=1}^{n} \mathcal{P}_{i,j} F_i \quad (\text{Fundamental System})
\]

In the lucky case, we can solve this system explicitly, but at any rate, we can use it iteratively to find a series expansion in \( q \). In either case, the desired weight-enumerator is given by

\[
\sum_{j \in E^{ni \times \text{sh}}} F_j(q; 1, \ldots, 1)
\]

Note that the variables \( x_1, \ldots, x_i \) corresponding to the \( l_i \)-parameter vertex type \( i \), for \( i = 1, \ldots, n \), serve as catalysts, all to be discarded (i.e. substituted by 1) at the end of the “reaction”.
I propose to use this set-up to study important subsets and supersets of ‘hard-to-count’ combinatorial families that arise in statistical physics: lattice animals, self-avoiding walks and polygons, and percolation. Here the number of components in each vertical slice is bounded, but the size itself is indefinitely large. So this can capture many more creatures than the finite-transfer matrix methods in which the creatures are confined to a strip. Even more interesting is the kind of functional equations for the generating functions of these sophisticated toy models. Recall that in the finite transfer-matrix method, one always gets rational functions making the critical behavior clearly unphysical and mathematically boring. The kind of functional equations that one should get now resemble those of the Renormalization Group method, and should give ‘interesting’, and hopefully closer-to-reality, critical behavior.

A General Theory Of Permutation Statistics

The study of permutations is very ancient, and was pursued in diverse cultures. For example, the Hebrew Book of Creation, Sefer Yetzira, that tradition attributes to the patriarch Abraham, and that was compiled c. 300 AD, lists the number of permutations of n objects for 2 ≤ n ≤ 7. The general formula, n!, and its, completely rigorous, inductive proof, can be found in “Sefer Ma’asei Khosev” written in the 14th century by Rabbi Levi Ben Gerson.

The more refined counting of permutations, according to various statistics, was probably started by Netto, and reached a climax with the monumental work of MacMahon. In more recent times, it was taken up by Foata and Schützenberger. It was Foata who coined the name ‘statistics’ in this context, as well as the name ‘major index’. It is nowadays a flourishing part of enumerative and algebraic combinatorics, with major contributions coming from: Anders Björner, Jennifer Galovich, Adriano Garsia, Kevin Kadell, Don Rawlings, Jeff Remmel, Bruce Sagan, Richard Stanley, Dennis Stanton, Dennis White, Michelle Wachs, Xavier (Gérard) Viennot and many others.

Dominique Foata and I plan to work on a general theory of permutation statistics that would be analogous to going from specific functions (like \( \cos(x), e^x, x^2 \)) to general analysis on functions. We would like to understand what makes the classical statistics ‘special’. For example, what are the conditions, for two different permutation statistics to be equidistributed, i.e. possess the same generating function? (like ‘inv’, the number of inversions, and the so-called major index, ‘maj’.)

Under what conditions is the generating function

\[
F_n(q) := \sum_{\pi \in S_n} q^{\text{stat}(\pi)},
\]

a) Closed form? (like in the case of \( \text{inv} \) and \( \text{maj} \)); b) ‘q-P-recursive’ (q-holonomic)? (i.e. \( F_n(q) \) satisfies a linear recurrence equation with coefficients that are polynomials in \((q, q^2)\). c) When is the global exponential generating function \( \sum_{n=0}^{\infty} \frac{F_n(q)}{n!} z^n \) nice?, like in the case of ‘des’ (when one gets the Eulerian polynomials, who are not ‘nice’ by themselves, but which possess a nice generating function.)
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What can one say about joint-distributions:

\[ F_n(q_1, \ldots, q_k) := \sum_{\pi \in S_n} q_1^{\text{stat}_1(\pi)} \cdots q_k^{\text{stat}_k(\pi)} \]

In a very interesting recent paper, Eric Babson and Einer Steingrimsson [BS] initiated their own theory of ‘pattern-statistics’ that include most of the known permutation statistics. They made several conjectures that Foata and I proved, by combining symbolic computation and ‘human combinatorics’. We hope that our ideas, combined with Babson and Steingrimsson’s seminal concept will prove useful.
REFERENCES


[BP] David Bressoud and James Propp, How the alternating sign conjecture was solved, Notices Amer. Math. Soc. 46 (1999), 637-646.


Biographical Sketch-Doron Zeilberger, P.I.

a. Professional Preparation

University of London, Mathematics, B.Sc. (With First Class Honours), 1972.

Weizmann Institute of Science, Mathematics, Ph.D., 1976.


b. Appointments


1978-1979: Georgia Institute of Technology, Visiting Assistant Professor.
c. (i) Five Relevant Publications


c. (ii) Five Other Publications


d. Synergetic Activities

The \textit{Wilf-Zeilberger Algorithmic Proof Theory} has been widely used by mathematicians and scientists alike, and has been implemented in all major computer algebra systems. Currently, the National Institute of Standards and Technology is ‘wiring’ the classic handbook of mathematical functions (the most widely cited book in science), by using WZ theory as its driving force.

My many Maple packages, in addition to doing the specific tasks that they were designed to do, when taken together, constitute a whole ‘research methodology’ for doing computer-assisted and computer-generated research.

e. Collaborators and Advisor

e(i). Recent Collaborators

Tewodros Amdeberhan (deVry Inst.), Shalosh B. Ekhad (Temple Univ.), Dominique Foata (Univ. of Strasbourg), Christian Krattenthaler (Univ. of Vienna), Istvan Nemes (RISC-Linz), John Noo-
nan (Mount Vernon Nazarene College), Marko Petkovsek (University of Slovenia), and Herb Wilf (University of Pennsylvania).

**e(ii). Graduate Advisor**

Harry Dym (Weizmann Institute).

**e(iii). Thesis Advisor**

So far, ten students received their Ph.D. degree under by supervision.


* Ethan Lewis (IBM, Israel) 1994.

* Craig Orr (NSA), 1994.


* Tewodros Amdeberhan (DeVry Inst. of Tech., NJ), 1997.


* Aaron Robertson (Colgate Univ., Hamilton, NY), 1999.

* Akalu Tefara (Grand Valley State Univ., MI), 2000.

* Anne Edlin (Holy Family College, NJ), 2000.

**Short Biography-Dominique Foata, consultant**

Dominique Foata received his *doctorat de état* under Marcel-Paul Schützenberger in 1965. He is currently distinguished professor of mathematics at the University of Strasbourg. His many honors include ICM 83’, and the UAP prize, 1985.