A Sharp Upper Bound for the Order of The Recurrence Outputted by Zeilberger's Algorithm 1

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Notation: q is an indeterminate (that commutes with everything). In this paper only, [a!] is short for (1 − q) ... (1 − q^n), and [a]_k is short for (1 − q)(1 − q^a + 1) ... (1 − q^a+k−1). The ground field is Q(q). The dependence on q is always understood.

q-Theorem. Let

$$F(n, k) = POL(q^n, q^k) \cdot H(n, k) \quad \text{(qProperHypergeometric)}$$

where POL(q^n, q^k) is a Laurent polynomial in (q^n, q^k), and

$$H(n, k) = \frac{\prod_{j=1}^{A}[a_j n + a_j k + a_j^0]!}{\prod_{j=1}^{C}[c_j n + c_j k + c_j^0]!} \cdot \sum_{j=1}^{B}[b_j n - b_j k + b_j^0]! \cdot \sum_{j=1}^{D}[d_j n - d_j k + d_j^0]! \cdot q^{jk(k-1)/2} z^k \quad \text{ (qPureHypergeometric)}$$

where the a_j, a^0_j (1 ≤ j ≤ A), b_j, b^0_j (1 ≤ j ≤ B), c_j, c^0_j (1 ≤ j ≤ C), d_j, d^0_j (1 ≤ j ≤ D) are non-negative integers, and z, a^0_j (1 ≤ j ≤ A), b^0_j (1 ≤ j ≤ B), c^0_j (1 ≤ j ≤ C), d^0_j (1 ≤ j ≤ D) are indeterminates, and J is an integer. We also assume that the factorization in (ProperHypergeometric) is maximal, i.e. POL(q^n, q^k) is as large as possible. This entails that the difference between any of the affine-linear expressions at the top and any of those at the bottom, is never a non-negative integer. Also let

$$L = \max \left( J + \sum_{j=1}^{A} a_j^2, \sum_{j=1}^{C} c_j^2 \right) + \max \left( -J + \sum_{j=1}^{D} d_j^2, \sum_{j=1}^{B} b_j^2 \right) \quad \text{ (qZBound)}$$

There exist polynomials in q^n, e_0(q^n),..., e_L(q^n), not all zero, and a rational function R(q^n, q^k) such that G(n, k) := R(q^n, q^k)F(n, k) satisfies

$$\sum_{i=0}^{L} e_i(q^n)F(n+i, k) = G(n, k+1) - G(n, k) \quad \text{ (qZpair)}$$

Proof: Let

$$\overline{H}(n, k) = \frac{\prod_{j=1}^{A}[a_j n + a_j k + a_j^0]!}{\prod_{j=1}^{C}[c_j n + c_j k + c_j^0]!} \cdot \sum_{j=1}^{B}[b_j n - b_j k + b_j^0]! \cdot \sum_{j=1}^{D}[d_j n - d_j k + d_j^0]! \cdot q^{jk(k-1)/2} z^k \quad \text{ (qZBound)}$$

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\[ f(k) = zq^k \prod_{j=1}^{A} (a_j n + a_j k + a_j'' + 1) \prod_{j=1}^{D} (d_j n - d_j k + d_j' + d_j'') + d_j L - d_j + 1), \]

and

\[ g(k) = \prod_{j=1}^{B} (b_j n - b_j k + b_j'' + 1) \prod_{j=1}^{C} (c_j n + c_j k + c_j' + c_j'') + c_j L + 1) \prod_{j=1}^{D} (d_j n - d_j k + d_j' + d_j'') + d_j L - d_j + 1), \]

Note that \( \overline{H}(n, k + 1)/\overline{H}(n, k) = f(k)/g(k) \). Write

\[ G(n, k) = g(k - 1)X(k)\overline{H}(n, k). \]

Substituting into (Zpair) and dividing both sides by \( \overline{H}(n, k) \), shows that it is equivalent to

\[ f(k)X(k + 1) - g(k - 1)X(k) - h(q^k) = 0, \quad \text{(qGosper)} \]

where

\[ h(q^k) := \sum_{i=0}^{L} c_i(n)POL(q^n q^i, q^k) \times \frac{H(n + i, k)}{\overline{H}(n, k)}. \]

Note that \( h(q^k) \) is a Laurent polynomial (in \( q^k \)) since

\[ \frac{H(n + i, k)}{\overline{H}(n, k)} = \prod_{j=1}^{A} (a_j n + a_j k + a_j'' + 1) \prod_{j=1}^{B} (b_j n - b_j k + b_j'' + 1) \prod_{j=1}^{C} (c_j n + c_j k + c_j' + c_j'' + 1) (L - i) c_j' \prod_{j=1}^{D} (d_j n - d_j k + d_j' + d_j'' + id_j + 1) (L - i) d_j'. \]

Let

\[ M_1 := -ldeg(h) - \max(-ldeg(f), -ldeg(g)) \quad \text{and} \quad M_2 := \deg(h) - \max(\deg(f), \deg(g)). \]

We claim that (qGosper) can always be solved (non-trivially) with \( X(k) \) a Laurent polynomial of \( q^k \) of low-degree \(-M_1\) and degree \( M_2 \). Writing

\[ X(k) = \sum_{i=-M_1}^{M_2} x_i(n)(q^k)^i, \]

substituting into (qGosper), and setting all the coefficients to 0, yields \(-ldeg(h) + \deg(h) + 1\) homogeneous linear equations for the \( M_1 + M_2 + L + 2 \) unknowns \( e_0(n), \ldots, e_L(n) \), and \( x_{-M_1}(n), \ldots, x_{M_2}(n) \). For such a not-all-zero solution to exist, we need \# unknowns - \# equations \(-1 \geq 0\), i.e. \( (M_1 + M_2 + L + 2) - (-ldeg(h) + \deg(h) + 1) - 1 \geq 0 \), i.e. \( L \geq \max(\deg(f), \deg(g)) + \max(-ldeg(f), -ldeg(g)). \)

But

\[ \deg(f) = J + \sum_{j=1}^{A} a_j^2, \quad -ldeg(f) = -J + \sum_{j=1}^{D} d_j, \quad \deg(g) = \sum_{j=1}^{C} c_j^2, \quad -ldeg(g) = \sum_{j=1}^{B} b_j^2. \]
This concludes the proof except that we did not rule out the possibility of \( e_0(n), \ldots, e_L(n) \) being all zero (all we are guaranteed, so far, is that it is not possible for all of \( e_0(n), \ldots, e_L(n) \), and \( x_0(n), \ldots, x_M(n) \) to be zero). But if all the \( e_i(n) \)'s are zeros, then \( h(k) \) is zero and (Gosper) becomes

\[
\frac{X(k + 1)}{X(k)} = \frac{g(k - 1)}{f(k)}.
\]

Since \( X(k) \) is a Laurent polynomial in \( q^k \), it means that the roots of \( f(k) = 0 \) differ from the roots of \( g(k - 1) = 0 \) by fixed non-negative integers, which is not possible because of the maximality hypothesis about \( POL(q^n, q^k) \). Note that the maximality hypothesis always holds, automatically, whenever we have the generic situation with the \( a_j^{''}, b_j^{''}, c_j^{''}, d_j^{''} \) arbitrary symbols. □