

**Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms
and the
Sharpening of Wilf-Zeilberger Theory**

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q-Notation. For k integer, $[a]_k := (1-q^a)(1-q^{a+1}) \dots (1-q^{a+k-1})$, if $k \geq 0$ and $[a]_k := 1/[a+k]_{-k}$ if $k < 0$. In order to avoid too many subscripts, we will denote $[a]_k$ by $qRF(a, k)$.

A Multi-Variate q-Zeilberger Algorithm

q-Theorem. Let

$$F(n; k_1, \dots, k_r) = POL(n; k_1, \dots, k_r) \cdot H(n; k_1, \dots, k_r) \quad , \quad (qMultiProperHypergeometric)$$

where $POL(n; k_1, \dots, k_r)$ is a Laurent polynomial in $(q^n, q^{k_1}, \dots, q^{k_r})$, and

$$H(n; k_1, \dots, k_r) = \frac{\prod_{j=1}^A qRF(a''_j, a'_j n + \sum_{i=1}^r a_{ji} k_i)}{\prod_{j=1}^C qRF(c''_j, c'_j n + \sum_{i=1}^r c_{ji} k_i)} \cdot q^{Q(n; k_1, \dots, k_r)} \cdot \prod_{i=1}^r z_i^{k_i} \quad , \quad (qMultiPureHypergeometric)$$

where the a'_j, c'_j are *non-negative* integers and the a_{ji}, c_{ji} are *integers*, while a''_j, c''_j and z_1, \dots, z_r are *commuting indeterminates*, and $Q(n; k_1, \dots, k_r)$ is a quadratic form in (n, k_1, \dots, k_r) . Then there exists an integer L , to be *explicitly* constructed in the course of the proof, polynomials in q^n , $e_0(q^n), e_1(q^n), \dots, e_L(q^n)$, *not all zero*, and r rational functions of $(q^n, q^{k_1}, \dots, q^{k_r})$, $R_I(n; k_1, \dots, k_r)$ ($I = 1, \dots, r$) such that

$$G_I(n; k_1, \dots, k_r) := R_I(n; k_1, \dots, k_r) F(n; k_1, \dots, k_r)$$

satisfy

$$\sum_{i=0}^L e_i(q^n) F(n+i, k) = \sum_{I=0}^r [G_I(n; k_1, \dots, k_{I-1}, k_I+1, k_{I+1}, \dots, k_r) - G_I(n; k_1, \dots, k_r)] \quad . \quad (qZtuple)$$

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