A WZ PROOF OF RAMANUJAN'S FORMULA FOR π
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Dedicated to Archimedes on his 2300th birthday

Archimedes computed π very accurately. Much later, Ramanujan discovered several infinite series for 1/π that enables one to compute π even more accurately. The most impressive one is ([Ra]):

\[
\frac{1}{\pi} = 2\sqrt{2} \sum_{k=0}^\infty \frac{(1/4)_k (1/2)_k (3/4)_k}{k!^3} (1103 + 26390k)(1/99)^{4k+2}.
\] (1)

This formula is an example of a non-terminating hypergeometric series identity. Many times, non-terminating series are either limiting cases or "analytic continuations" of terminating identities, which are now known to be routinely provable by computer. [WZ].

While we do not know of a terminating generalization of (1), we do know how to give a WZ proof of another formula for π, also given by Ramanujan [Ra], and included in his famous letter to Hardy. This formula is:

\[
\frac{2}{\pi} = \sum_{k=0}^\infty (-1)^k (4k+1) \frac{(1/2)_k^2}{k!^3}.
\] (2)

The terminating version, that we will prove is

\[
\frac{\Gamma(3/2 + n)}{\Gamma(3/2)\Gamma(n + 1)} = \sum_{k=0}^\infty (-1)^k (4k+1) \frac{(1/2)_k^2 (-n)_k}{k!^2 (3/2 + n)_k}.
\] (3)

To prove it for all positive integers n, we call the summand divided by the left side \( F(n,k) \), and cleverly construct

\[
G(n,k) := \frac{(2k+1)^2}{(2n+2k+3)(4k+1)} F(n,k),
\]

with the motive that \( F(n+1,k) - F(n,k) = G(n,k) - G(n,k-1) \) (check!), and summing this last identity w.r.t \( k \) shows that \( \sum_k F(n,k) \equiv Constant \), which is seen to be 1, by plugging in \( n = 0 \). This proves (3). To deduce (2), we "plug" in \( n = -1/2 \), which is legitimate in view of Carlson’s theorem [Ba].

REFERENCES


1 Department of Mathematics, Temple University, Philadelphia, PA 19122. [ekhad, zeilberg]@math.temple.edu; http://www.math.temple.edu/~ekhad, zeilberg]. The work of the second author was supported in part by the NSF. This paper was published in p.107-108 of 'Geometry, Analysis, and Mechanics', ed. by J. M. Rassias, World Scientific, Singapore 1994.