

## A WZ PROOF OF RAMANUJAN'S FORMULA FOR $\pi$

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*Dedicated to Archimedes on his 2300<sup>th</sup> birthday*

Archimedes computed  $\pi$  very accurately. Much later, Ramanujan discovered several infinite series for  $1/\pi$  that enables one to compute  $\pi$  even more accurately. The most impressive one is ([Ra]):  $((a)_k$  denotes, as usual,  $a(a+1)\dots(a+k-1)$ .)

$$\frac{1}{\pi} = 2\sqrt{2} \sum_{k=0}^{\infty} \frac{(1/4)_k (1/2)_k (3/4)_k}{k!^3} (1103 + 26390k) (1/99)^{4k+2}. \quad (1)$$

This formula is an example of a *non-terminating* hypergeometric series identity. Many times, non-terminating series are either limiting cases or "analytic continuations" of *terminating identities*, which are now known to be routinely provable by computer. [WZ].

While we do not know of a terminating generalization of (1), we do know how to give a WZ proof of another formula for  $\pi$ , also given by Ramanujan [Ra], and included in his famous letter to Hardy. This formula is:

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{(1/2)_k^3}{k!^3}. \quad (2)$$

The terminating version, that we will prove is

$$\frac{\Gamma(3/2+n)}{\Gamma(3/2)\Gamma(n+1)} = \sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{(1/2)_k^2 (-n)_k}{k!^2 (3/2+n)_k}. \quad (3)$$

To prove it for all *positive* integers  $n$ , we call the summand divided by the left side  $F(n, k)$ , and cleverly construct

$$G(n, k) := \frac{(2k+1)^2}{(2n+2k+3)(4k+1)} F(n, k),$$

with the motive that  $F(n+1, k) - F(n, k) = G(n, k) - G(n, k-1)$  (check!), and summing this last identity w.r.t  $k$  shows that  $\sum_k F(n, k) \equiv \text{Constant}$ , which is seen to be 1, by plugging in  $n=0$ . This proves (3). To deduce (2), we "plug" in  $n=-1/2$ , which is legitimate in view of Carlson's theorem [Ba].

## REFERENCES

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