Proof of the Riemann Hypothesis, and some other Hitherto Undemonstrated Propositions

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Introduction

One suggestion by Pierre Cartier (see Ruelle's "Chance and Chaos" p. 193) is that the axioms of set theory are actually inconsistent, but that a proof of contradiction would be so long that it could not be performed in our physical universe!

While the consistency of the axioms of set theory remains open, in this note I announce, and sketch the proofs, of several hitherto open problems. The proofs follow the celebrated WZ methodology, suitably extended, that reduces the truth of statements to routinely verifiable *certificates.* Fortunately, the lengths of these certificates fit very comfortably, even in our very own planet. Unfortunately they far exceed the hard-disk allocation of 30MB of my ftp and http directories. The application for enlarged allocation is still pending.

On the other hand, perhaps the actual certificates are besides the point. John Littlewood even had a name for those pedants who insist on seeing the certificates and carrying out the purely routine algebra: *obtuse.* In fact, this is a good way to find out who is obtuse and who is not. Once my application for extended disk-space is granted, I will take a record of all those who try to download the certificates, and this will be used to compile a *Directory of Obtuse Mathematicians*, that would be viewable for a modest fee, that will help defray the cost of maintaining the ftp facility.

Two Proofs of the Riemann Hypothesis

"Elementary Proof"

Let $\psi(x) := \sum_{p^m \leq x} \log p$. Tchebychev proved that, for large x, $A_1 x \leq \psi(x) \leq A_2 x$, by using the ridiculously simple recurrence:

$$(\log 2) \cdot x + O(log(x)) = \psi(x) - \psi(x/2) + \psi(x/3) - \psi(x/4) + \ldots \quad ,$$

that yields $A_1 = \log 2$ and $A_2 = 2 \log 2$.

This was considerably improved by Tchebychev himself and James Joseph Sylvester, who found other, more complicated, but still linear, recurrences, that brought A_1 up and A_2 down.

The next step was realized by Erdös and Selberg, who combined two simple recursive inequalities for $R(x) := \psi(x) - x$. The first one is linear (in $|R(x)|$):

$$|R(x)| \leq \frac{1}{\log x} \sum_{n \leq x} |R(x/n)| + O(\frac{x \log \log x}{\log x}) \quad ,$$

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while the second one is quadratic:

$$\sum_{n\le x} \frac{\log n}{n} R(n) = -\sum_{n\le x} \frac{1}{n} R(n)R(x/n) + O(x) \quad ,$$

from which it follows that $|R(x)| \le \sigma x$ for $x > x_\sigma$, for every $\sigma > 0$.

Our enhancement of the Erdös-Selberg method is related to it in the same way as it is related to the Tchebychev-Sylvester method.

By using *advanced sieving*, we derive, much longer, *super-holonomic* recursive inequalities that immediately imply that $\pi(x) - li(x) = O(x^{1/2+\varepsilon})$. These inequalities will be available from my ftp and http site mentioned above, in compressed, packed, and gunzipped form, occupying 2000 MB. □

'Analytic' Proof of RH

The first proof is always the hardest. After we found the above proof, we have also found another, slightly shorter (1000 MB) proof. The first step was undertaken by Jensen, who showed that RH is equivalent to the following inequality:

$$\Delta(x,y) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(t)\Phi(s)e^{i(t+s)x}e^{(t-s)y}(t-s)^2 dtds \ge 0 \quad ,$$

where

$$\Phi(t) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{2t} - 3\pi n^2 e^{5t})e^{-\pi n^2 e^{4t}} \quad .$$

It follows from the super-WZ theory that $\Delta(x,y)$ is super-holonomic. It follows from Hilearnie's 17^{th} problem that if it is indeed non-negative, then it should be possible to write it as a sum of squares of super-holonomic functions. This indeed turned out to be the case. We exhibit these explicitly, by giving the super-holonomic recurrences each of these (≤ 100000) functions satisfy. □

P ≠ NP

A.Razborov introduced *legitimate* lattices in order to obtain lower bounds for the monotone circuit complexity of CLIQUE. By using very naive 'counting' (sometimes called, erroneously, 'probabilistic') methods, he found super-polynomial lower bounds. These were upgraded to 'exponential' (in $n^{1/6}$) by Andreev, and N. Alon[2] and R.Boppana.

In order to tackle general circuits, we introduce the revolutionary concept of *Kosher* lattice, and then exhibit a specific, albeit very large, such lattice. By using *advanced sieving* once again we get the desired super-polynomial lower bounds for the (generalized) distance of the Boolean function

[2] N. Alon is a thinly disguised pseudonym of the brilliant combinatorialist A. Nilli.

describing CLIQUE and our Kosher lattice. Both the description of our Kosher lattice, and the rather long inequalities implying the bounds will be soon available at the above-mentioned site. □

(Added in Proof:) N. Alon and A. Nilli have informed us that both our lattices and inequalities are unnecessarily long, and that they are preparing a 1-page short proof, using a slight modification of our Kosher lattices, aptly named *Glatt Kosher lattices.* Their proof is entirely self-contained, and only uses polynomials over finite fields, and the trivial fact that a polynomial can't have more zeros than its degree. Due to this development, and as a concession to our system administrator, we decided *not* to include the original lattice and inequalities at our ftp site.

Lehmer's Conjecture : Ramanujan's $\tau(n)$ is Never Zero

The key-idea was to realize that the statement $\tau(n) \neq 0$ is equivalent to $\tau(n)^2 > 0$. The proof now follows the same lines as the second proof of RH. $\tau(n)$, and hence $\tau(n)^2$, belongs to another generalization of the class of holonomic sequences, called $\theta-$holonomic sequences. As expected, $\tau(n)^2$ is exhibitable as a sum of squares of such sequences. It turned out that ≤ 10000 sequences, the description of each requiring $\leq 3KB$, suffice. Finally, it could be easily verified that they cannot vanish simultaneously. □

Wiles's Theorem

Lest I would be accused of only solving open problems, I will also outline a new proof of Wiles's theorem. Unlike the original proof, that can only be understood and verified by those that are referenced in Wiles's paper, the present proof can be understood and verified by any obtuse mathematician.

Let $W(a,b,c,n) := (a^n + b^n - c^n)^2$. We have to prove that when $a,b,c > 0$ and $n > 2$, $W(a,b,c,n) > 0$. It is easy to find many recurrences (both linear and non-linear) satisfied by $W(a,b,c,n)$. Amongst those, there is at least one, of the form,

$$W(a,b,c,n) = F(n-3 \cup \{W(a-\alpha, b-\beta, c-\gamma, n-m) | 0 < \alpha+\beta+\gamma+m \leq 10000\}) \quad ,$$

where F is an explicit rational function of $\binom{10004}{4} - 1$ variables, with both numerator and denominator polynomials of (total) degree ≤ 100000, and most importantly, with *positive* coefficients. Wiles's theorem now follows by induction. □