

Sharp Upper Bounds for the Orders of The Recurrences Outputted by the Zeilberger and q-Zeilberger Algorithms

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“Er muß sozusagen die Leiter wegwerfen, nachdem er auf ihr hinaufgestiegen ist”

—L. Wittgenstein (Logisch-philosophische Abhandlung, 6.54)

Abstract: We do what the title promises, and as a bonus we get much simplified versions of these algorithms, that do not make any explicit mention of Gosper’s algorithm.

Notation. For k integer, $(z)_k := z(z+1)\dots(z+k-1)$, $[a]_k := (1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1})$, if $k \geq 0$ and $(z)_k := 1/(z+k)_{-k}$, $[a]_k := 1/[a+k]_{-k}$, if $k < 0$. For a *Laurent polynomial* $p(t)$ of t , $\deg(p)$ is the *degree*, and $\text{ldeg}(p)$ is the *low-degree*, e.g., if $p = 4t^{-3} + 2t^{-1} + 4 + 3t + t^2$, $\deg(p) = 2$, $\text{ldeg}(p) = -3$.

Theorem. Let

$$F(n, k) = POL(n, k) \cdot H(n, k) \quad , \quad (\text{ProperHypergeometric})$$

where $POL(n, k)$ is a polynomial in (n, k) and

$$H(n, k) = \frac{\prod_{j=1}^A (a''_j)_{a'_j n + a_j k} \prod_{j=1}^B (b''_j)_{b'_j n - b_j k}}{\prod_{j=1}^C (c''_j)_{c'_j n + c_j k} \prod_{j=1}^D (d''_j)_{d'_j n - d_j k}} z^k \quad , \quad (\text{PureHypergeometric})$$

where the $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are *non-negative integers*, and $z, a''_j, b''_j, c''_j, d''_j$ are *commuting indeterminates*. We also assume that the factorization in *(ProperHypergeometric)* is *maximal*, i.e. $POL(n, k)$ is of the largest possible degree. Let

$$L = \max \left(\sum_{j=1}^A a_j + \sum_{j=1}^D d_j \quad , \quad \sum_{j=1}^B b_j + \sum_{j=1}^C c_j \right) \quad . \quad (\text{ZBound})$$

There exist polynomials in n , $e_0(n), \dots, e_L(n)$, *not all zero*, and a rational function $R(n, k)$ such that $G(n, k) := R(n, k)F(n, k)$ satisfies

$$\sum_{i=0}^L e_i(n)F(n+i, k) = G(n, k+1) - G(n, k) \quad . \quad (\text{Zpair})$$

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Furthermore, *in general*, L cannot be made any smaller, i.e. the inequality $Order \geq L$ is *sharp*.

Proof: Let

$$\overline{H}(n, k) = \frac{\prod_{j=1}^A (a_j'')_{a_j' n + a_j k} \prod_{j=1}^B (b_j'')_{b_j' n - b_j k}}{\prod_{j=1}^C (c_j'')_{c_j' (n+L) + c_j k} \prod_{j=1}^D (d_j'')_{d_j' (n+L) - d_j k}} z^k \quad ,$$

$$f(k) = z \prod_{j=1}^A (a_j' n + a_j k + a_j'')_{a_j} \prod_{j=1}^D (d_j' (n+L) - d_j k + d_j'' - d_j)_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^B (b_j' n - b_j k + b_j'' - b_j)_{b_j} \prod_{j=1}^C (c_j' (n+L) + c_j k + c_j'')_{c_j} \quad .$$

Note that $\overline{H}(n, k+1)/\overline{H}(n, k) = f(k)/g(k)$. Write

$$G(n, k) = g(k-1)X(k)\overline{H}(n, k) \quad . \quad (Ansatz)$$

Substituting into (*Zpair*) and dividing both sides by $\overline{H}(n, k)$, shows that it is equivalent to

$$f(k)X(k+1) - g(k-1)X(k) - h(k) = 0 \quad , \quad (Gosper)$$

where

$$h(k) := \sum_{i=0}^L e_i(n) POL(n+i, k) \cdot \frac{H(n+i, k)}{\overline{H}(n, k)} \quad .$$

Note that $h(k)$ is a *polynomial* since

$$\frac{H(n+i, k)}{\overline{H}(n, k)} = \prod_{j=1}^A (a_j' n + a_j k + a_j'')_{i a_j'} \prod_{j=1}^B (b_j' n - b_j k + b_j'')_{i b_j'} \prod_{j=1}^C (c_j' n + c_j k + c_j'' + i c_j')_{(L-i) c_j'} \prod_{j=1}^D (d_j' n - d_j k + d_j'' + i d_j')_{(L-i) d_j'} \quad .$$

We claim that (*Gosper*) can always be solved (non-trivially) with $X(k)$ being a polynomial of degree $M := deg(h) - \max(deg(f), deg(g))$. Writing

$$X(k) = \sum_{i=0}^M x_i(n) k^i \quad , \quad (Ansatz1)$$

substituting into (*Gosper*), and setting all the coefficients to 0, yields $deg(h) + 1$ *homogeneous* linear equations for the $M + L + 2$ unknowns $e_0(n), \dots, e_L(n)$, and $x_0(n), \dots, x_M(n)$. For such a *not-all-zero* solution to exist, we need $\# unknowns - \# equations - 1 \geq 0$, i.e. $(M + L + 2) - (deg(h) + 1) - 1 \geq 0$, i.e. $L \geq \max(deg(f), deg(g))$. But

$$deg(f) = \sum_{j=1}^A a_j + \sum_{j=1}^D d_j \quad , \quad deg(g) = \sum_{j=1}^B b_j + \sum_{j=1}^C c_j \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \dots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \dots, e_L(n)$, and $x_0(n), \dots, x_M(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(k)$ is zero and (*Gosper*) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} \quad .$$

Since $X(k)$ is a polynomial, it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(n, k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a''_j, b''_j, c''_j, d''_j$ arbitrary (commuting) *symbols*.

To prove that (*Zbound*) is sharp, take $F(n, k) = 1/((1)_k(1)_{n-k})$ and note that L cannot be 0, since otherwise it would have been gosperable with respect to k , but it is not, as can be seen by performing the Gosper algorithm[G] on it. \square

q-Theorem. Let

$$F(n, k) = POL(q^n, q^k) \cdot H(n, k) \quad , \quad (q\text{ProperHypergeometric})$$

where $POL(q^n, q^k)$ is a Laurent polynomial in (q^n, q^k) , and

$$H(n, k) = \frac{\prod_{j=1}^A [a''_j]_{a'_j n + a_j k} \prod_{j=1}^B [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^C [c''_j]_{c'_j n + c_j k} \prod_{j=1}^D [d''_j]_{d'_j n - d_j k}} q^{Jk(k-1)/2} z^k \quad , \quad (q\text{PureHypergeometric})$$

where the $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are *non-negative integers*, and $z, a''_j, b''_j, c''_j, d''_j$ are *indeterminates*, and J is an integer. We also assume that the factorization in (*qProperHypergeometric*) is *maximal*, i.e. $POL(q^n, q^k)$ is as 'large' as possible. Let

$$L = \max \left(J + \sum_{j=1}^A a_j^2, \sum_{j=1}^C c_j^2 \right) + \max \left(-J + \sum_{j=1}^D d_j^2, \sum_{j=1}^B b_j^2 \right) \quad . \quad (q\text{ZBound})$$

There exist polynomials in $q^n, e_0(q^n), \dots, e_L(q^n)$, *not all zero*, and a rational function $R(q^n, q^k)$ such that $G(n, k) := R(q^n, q^k)F(n, k)$ satisfies

$$\sum_{i=0}^L e_i(q^n)F(n+i, k) = G(n, k+1) - G(n, k) \quad . \quad (q\text{Zpair})$$

Furthermore, *in general*, L , cannot be made any smaller, i.e. the inequality $Order \geq L$ is *sharp*.

Proof: Let

$$\overline{H}(n, k) = \frac{\prod_{j=1}^A [a''_j]_{a'_j n + a_j k} \prod_{j=1}^B [b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^C [c''_j]_{c'_j(n+L) + c_j k} \prod_{j=1}^D [d''_j]_{d'_j(n+L) - d_j k}} q^{Jk(k-1)/2} z^k \quad ,$$

$$f(k) = zq^{Jk} \prod_{j=1}^A [a'_j n + a_j k + a''_j]_{a_j} \prod_{j=1}^D [d'_j(n+L) - d_j k + d''_j - d_j]_{d_j} \quad ,$$

and

$$g(k) = \prod_{j=1}^B [b'_j n - b_j k + b''_j - b_j]_{b_j} \prod_{j=1}^C [c'_j(n+L) + c_j k + c''_j]_{c_j} \quad .$$

Note that $\overline{H}(n, k+1)/\overline{H}(n, k) = f(k)/g(k)$. Write

$$G(n, k) = g(k-1)X(k)\overline{H}(n, k) \quad . \quad (qAnsatz)$$

Substituting into (*Zpair*) and dividing both sides by $\overline{H}(n, k)$, shows that it is equivalent to

$$f(k)X(k+1) - g(k-1)X(k) - h(q^k) = 0 \quad , \quad (qGosper)$$

where

$$h(q^k) := \sum_{i=0}^L e_i(n) POL(q^n q^i, q^k) \cdot \frac{H(n+i, k)}{\overline{H}(n, k)} \quad .$$

Note that $h(q^k)$ is a *Laurent polynomial* (in q^k) since

$$\frac{H(n+i, k)}{\overline{H}(n, k)} =$$

$$\prod_{j=1}^A [a'_j n + a_j k + a''_j]_{i a'_j} \prod_{j=1}^B [b'_j n - b_j k + b''_j]_{i b'_j} \prod_{j=1}^C [c'_j n + c_j k + c''_j + i c'_j]_{(L-i) c'_j} \prod_{j=1}^D [d'_j n - d_j k + d''_j + i d'_j]_{(L-i) d'_j} \quad .$$

Let

$$M_1 := -ldeg(h) - \max(-ldeg(f), -ldeg(g)) \quad , \quad M_2 := deg(h) - \max(deg(f), deg(g)) \quad .$$

We claim that (*qGosper*) can always be solved (non-trivially) with $X(k)$ a Laurent polynomial of q^k of low-degree $-M_1$ and degree M_2 . Writing

$$X(k) = \sum_{i=-M_1}^{M_2} x_i(n) (q^k)^i \quad , \quad (qAnsatz1)$$

substituting into (*qGosper*), and setting all the coefficients to 0, yields $-ldeg(h) + deg(h) + 1$ *homogeneous* linear equations for the $M_1 + M_2 + L + 2$ unknowns $e_0(n), \dots, e_L(n)$, and $x_{-M_1}(n), \dots, x_{M_2}(n)$. For such a *not-all-zero* solution to exist, we need $\#unknowns - \#equations - 1 \geq 0$, i.e. $(M_1 + M_2 + L + 2) - (-ldeg(h) + deg(h) + 1) - 1 \geq 0$, i.e. $L \geq \max(deg(f), deg(g)) + \max(-ldeg(f), -ldeg(g))$. But

$$deg(f) = J + \sum_{j=1}^A a_j^2 \quad , \quad -ldeg(f) = -J + \sum_{j=1}^D d_j^2 \quad , \quad deg(g) = \sum_{j=1}^C c_j^2 \quad , \quad -ldeg(g) = \sum_{j=1}^B b_j^2 \quad .$$

This concludes the proof *except* that we did not rule out the possibility of $e_0(n), \dots, e_L(n)$ being all zero (all we are guaranteed, so far, is that it is not possible for *all* of $e_0(n), \dots, e_L(n)$, and $x_{-M_1}(n), \dots, x_{M_2}(n)$ to be zero). But if all the $e_i(n)$'s are zeros, then $h(q^k)$ is zero and (*qGosper*) becomes

$$\frac{X(k+1)}{X(k)} = \frac{g(k-1)}{f(k)} .$$

Since $X(k)$ is a Laurent polynomial in q^k , it means that the roots of $f(k) = 0$ differ from the roots of $g(k-1) = 0$ by *fixed* non-negative integers, which is not possible because of the maximality hypothesis about $POL(q^n, q^k)$. Note that the maximality hypothesis always holds, automatically, whenever we have the *generic* situation with z and the $a_j'', b_j'', c_j'', d_j''$ arbitrary *symbols*.

To prove that (*qZbound*) is sharp, take $F(n, k) = q^{k(k-1)/2} / ([1]_k [1]_{n-k})$, and note that L cannot be 0, since otherwise it would have been q-gosperable with respect to k , but it is not, as can be seen by performing the q-Gosper algorithm ([Kor][PR], or use qEKHAD) on it. \square

Comments

1. The bounds in (*ZBound*) and (*qZBound*) considerably improve those of [WZ] (Theorems 3.1 and 5.1, see also [PWZ] and [Koe]), that relied on Sister Celine's Technique, and, as we proved, are *sharp* for the *generic* case. However, sometimes a system of linear equations with more equations than unknowns *does* have a non-trivial solution, and also, sometimes one can find higher-degree polynomial solutions to (*Gosper*) and (*qGosper*), so in *specific* cases, it is possible to have recurrences of lower order. This is the case for all the non-trivial classical hypergeometric (binomial-coefficient) sums that admit a closed-form evaluation.

2. The proofs imply new, simplified, versions of the Zeilberger[Z1][Z2][PS] and q-Zeilberger[Kor][PR] algorithms. These new versions do not rely on Gosper's algorithm *explicitly*, but, of course, were inspired by it. In fact, they were designed by applying the Zeilberger and q-Zeilberger algorithms *once and for all*, to the *generic* cases. It so happens, that in this case, a simplified version of Gosper (and q-Gosper) suffices, and it is so simple that it can be incorporated implicitly. So old-Zeilberger (and hence Gosper and q-Gosper) is the Wittgensteinian ladder that we *must* throw away after we climbed it.

The *running-time* complexity of these new versions are comparable to the old versions, but their *program-length complexity* (in the sense of Chaitin-Kolmogorov) are considerably smaller

The simplified Zeilberger and q-Zeilberger algorithms, apply also to *specific*, non-generic summands. Start by taking $L = 0$ and try the *ansatzes* (*Ansatz*) and (*qAnsatz*), but with M (for the q-case: M_1, M_2) possibly larger than the ones in the theorem (which are determined by plugging them into (*Gosper*) or (*qGosper*), and equating the leading coefficient(s), and finding out whether they can vanish for integral M (or M_1, M_2). Then one solves the resulting set of linear equations. If there is no non-trivial solution, then, one increases L by 1, until success is reached. The theorems guarantee that eventually we will succeed, at worst, with the L 's given by (*ZBound*) and (*qZBound*).

3. These simplified versions are implemented in the Maple packages ZEILBERGER and qZEILBERGER available from <http://www.math.rutgers.edu/~zeilberg/Zeilberger/sharp.html>, where one can also find several examples.

4. The present article was intentionally written in a terse, unmotivated, style, in order to emphasize its simplicity and self-containedness. Readers who wish to see more motivation are welcome to look at an earlier version, that only covers the ordinary case, that is also available from the above webpage.

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References

[G] R. W. Gosper, Jr., *Decision procedures for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75** (1975), 40-42.

[GKP] R. L. Graham, D. E. Knuth, and O. Patashnik, "Concrete Mathematics", second edition, Addison Wesley, 1993.

[Koe] W. Koepf, "*Hypergeometric Summation*", Vieweg, 1998.

[Kor] T. H. Koornwinder, *On Zeilberger's algorithm and its q-analog*, J. Comp. Appl. Math. **48** (1993), 91-111.

[Pa] P. Paule, *Short and easy proofs of the Rogers-Ramanujan identities and of identities of similar type*, Electronic J. Combinatorics **1** (1994) #R10.

[PR] P. Paule and A. Riese, *A Mathematica q-analog of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping*, in: "Special Functions, q-series and related topics, 179-210, Fields Inst. Comm. **14**, Amer. Math. Soc., Providence, RI, 1997.

[PS] P. Paule and M. Schorn, *A Mathematica version of Zeilberger's algorithm for proving binomial coefficients identities*, J. Symbolic Comput. **20** (1995), 673-698.

[PWZ] M. Petkovsek, H. S. Wilf and D. Zeilberger, *A=B*, AK Peters, Wellesley, (1996). [available on-line from the authors' websites.]

[WZ] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108**, 575-633 (1992). [available on-line from the authors' websites.]

[Z1] D. Zeilberger, *A Fast Algorithm for proving terminating hypergeometric identities*, Discrete Math **80**, 207-211, (1990). [available on-line from the author's website.]

[Z2] D. Zeilberger, *The method of creative telescoping*, J. Symbolic Computation **11**, 195-204 (1991).