Experimenting with Standard Young Tableaux

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In fond memory of Herbert Wilf and Albert Nijenhuis, and in honor of Curtis Greene

Abstract: Using Symbolic Computation with Maple, we can discover lots of (rigorously-proved!) facts about Standard Young Tableaux, in particular the distribution of the entries in any specific cell, and the sorting probabilities.

Maple package

This article is accompanied by a Maple package, `SYT.txt`, available from:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/SYT.txt` .

The web-page of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/syt.html` ,

contains many input and output files, some of which will be referred to later.

Preface

One of the most *iconic* objects in mathematics, both *concrete* [K], and *abstract* [F], are Standard Young Tableaux [Wi]. Recall that an *integer partition*, or *partition*, for short, aka *shape*, of a non-negative integer n, with k **parts**, is a non-increasing sequence of **positive** integers

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \quad ,$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0 \quad ,$$

such that

$$\lambda_1 + \dots + \lambda_k \, = \, n \quad .$$

The **Ferrers diagram** (or **Young diagram**) of a partition λ is a left-justified array of dots (or empty boxes) where the top row has λ_1 dots (boxes), the second row has λ_2 boxes, ..., the k-th row has λ_k dots. For example, the Ferrers diagram of $(4, 4, 3, 1)$ is

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & \\ * & & & \end{array} \quad .$$

Given a shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ with n boxes, a **standard Young tableau** is a way of filling the boxes with the integers $\{1, \ldots, n\}$, such that each of them shows up (necessarily once) and both rows and columns are increasing. More formally, it is an array

$$T_{i,j} \quad , \quad 1 \leq i \leq k \quad , \quad 1 \leq j \leq \lambda_i \quad ,$$

such that $T_{i,j} < T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ whenever they exit.

Here are the five standard Young tableaux of shape $(2, 2, 1)$:

$$\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & \end{matrix} \quad , \quad \begin{matrix} 1 & 2 \\ 3 & 5 \\ 4 & \end{matrix} \quad , \quad \begin{matrix} 1 & 3 \\ 2 & 4 \\ 5 & \end{matrix} \quad , \quad \begin{matrix} 1 & 3 \\ 2 & 5 \\ 4 & \end{matrix} \quad , \quad \begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{matrix} \quad .$$

To see the set of standard Young tableaux of shape `L` in our Maple package, type `SYT(L);` . For example to see the above five tableaux type `SYT([2,2,1]);` .

The total number of standard Young tableaux of shape λ, denoted by f^λ, is famously given by the *hook length formula*, or equivalently (and more convenient for us) by the **Young-Frobenius formula** (see [K]).

$$f^\lambda = \frac{(\lambda_1 + \lambda_2 + \ldots + \lambda_k)!}{(\lambda_1 + k - 1)!(\lambda_2 + k - 2)! \cdots \lambda_k!} \cdot \prod_{i=1}^{k} \prod_{j=i+1}^{k} (\lambda_i - \lambda_j + j - i) \quad .$$

Fix a shape λ and fix a cell $[i, j]$, $1 \leq i \leq k, 1 \leq j \leq \lambda_i$.

Who can be the occupant of that cell?

Calling that occupant r, we have:

$$ij \leq r \leq \lambda_1 + \lambda_2 + \ldots + \lambda_{i-1} + \lambda'_1 + \lambda'_2 + \ldots + \lambda'_{j-1} - (i-1)(j-1) + 1 \quad ,$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ is the **conjugate partition**.

This gives a certain **probability distribution**. What is it?

For example, with the shape $\lambda = (2, 2, 1)$ again, and the cell $[2, 1]$, the set of possible occupants is $\{2, 3\}$, and the probability of it being 2 is $\frac{3}{5}$ and of it being 3 is $\frac{2}{5}$.

Later on we will be interested not in *specific* shapes, but in general (mostly rectangular) shapes, with a fixed number of rows, but arbitrary (i.e. *symbolic*) shape. Fixing the number of rows to be k (where k is *numeric*), and regarding the shape $(n, n, \ldots, n)$, where n is repeated k times, we would be interested in deriving closed-form expressions (as rational functions of n), for the probability distribution of the possible occupants of a given first row cell $[1, j]$, for any given **numeric** integer j. Note that the possible occupants of $[1, j]$ are $j, j + 1, j + 2, \ldots, k(j - 1) + 1$.

Once we found these expressions in n, we can ask about the *limiting distribution*, that Maple can find for us. Then we can also hope to see how it varies with j and look at the *meta-limiting* behavior as j gets larger.

Another kind of question, inspired by the beautiful work of Chan, Pak, and Panova [CPP1][CPP2], is to study the **sorting probabilities**. Given two cells c_1 and c_2, draw a standard Young tableau *uniformly at random*. Who is bigger?

The occupant of c_1 or the occupant of c_2?

Of course if the two cells are *related*, i.e. one of them, say c_2, is (weakly) below and (weakly) to the right of the other, c_1, i.e. in the underlying poset $c_1 < c_2$, then of course, *always* $T_{c_1} < T_{c_2}$, but what if they are **not** related i.e. writing

$$c_1 = [i_1, j_1] \quad , \quad c_2 = [i_2, j_2] \quad ,$$

we have $i_1 < i_2$ but $j_1 > j_2$.

The **sorting probability** is defined by:

$$SP(\lambda, c_1, c_2) := Pr(T_{c_1} > T_{c_2}) - Pr(T_{c_2} > T_{c_1}) = 2\,Pr(T_{c_1} > T_{c_2}) - 1 \quad ,$$

where T is a random standard Young tableau of shape λ.

In particular, following Chan-Pak-Panova, we are interested in the **minimal** (absolute value) of the sorting probabilities, over all pairs of cells, as the shapes get larger.

Going back to the shape $\lambda = (2,2,1)$, we see that for the first two tableaux the occupant of $(1,2)$ is less than the occupant of $(2,1)$, while for the last three ones it is the reverse. Hence the probability of $T_{1,2} < T_{2,1}$ is $\frac{2}{5}$, and so $SP(221, [1,2], [2,1]) = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$. The **minimal sorting probability** for that shape is also $\frac{1}{5}$ (check!)

Simulation

One way to answer these questions, *approximately*, is via *simulation*. The amazing Greene-Nijenhuis-Wilf [GrNW] algorithm (that also lead to a beautiful probabilistic proof of the *hook lenghth formula*) inputs a shape, λ, and outputs, *uniformly at random*, one of the f^λ standard Young tableaux of that shape. By sampling many of them, we can get approximations to the quantities of interest.

Procedure `GNW(L)` in our Maple package `SYT.txt` implements the Greene-Nijenhuis-Wilf algorithm. For example, try

```
GNW([4,3,2]);   ,
```

in order to get, uniformly at random, one of the 168 standard Young tableaux of shape $(4,3,2)$.

To get approximations for the probability generating function, using the variable `x`, of the distribution of the occupants of cell `c` in a random Young tableau of shape `L`, by sampling `K` random tableaux, type

```
SiOcGF(L,c,x,K);
```

For example for the shape $(4,4,4)$ and the cell $[2,2]$, with 10000 tries, type: `SiOcGF([4,4,4],[2,2],x,10000);`

getting something like (of course it is slightly different each time)

$$.1090000000\,x^7 + .2875000000\,x^6 + .3639000000\,x^5 + .2396000000\,x^4 \quad .$$

To get approximations for the sorting probability of cell `c1` vs. cell `c2`, in the shape `L`, by sampling K tableaux: enter:

```
SiPr(L,c1,c2,K);
```

For example,

```
SiPr([3,3,3],[1,2],[2,1],10000);
```

would give something like `0.010400000`. Of course, in this particular case the exact answer is obviously zero, by symmetry, so getting something close to 0 is a good *sanity check*.

Symbol Crunching in order to find The Probability Distribution of the Occupants of a Specific Cell in a Symbolic Shape

For the sake of exposition, we will mostly be concerned with standard Young tableaux of rectangular shape. Fix the number of rows k, and consider the shape

$$(n, n, \dots, n) \quad ,$$

where n is repeated k times. More generally all our algorithms carry over to the general *symbolic* shape

$$(n_1, n_2, \dots, n_k) \quad ,$$

where $n_1 \geq n_2 \geq n_k \geq 0$, and they are all left symbolic. Using Young-Frobenius we get an **explicit** formula for their total number in the form of a certain **rational function**, of $(n_1, \dots, n_k)$ times the **multinomial coefficient**

$$\frac{(n_1 + \ldots + n_k)!}{n_1! \cdots n_k!} \quad .$$

In the special case of a $k \times n$ rectangular shape: (i.e. $(n, \dots, n)$, where n is repeated k times), it is a certain rational function of n (namely $\frac{(k-1)!}{(n+1)_1 (n+1)_2 \cdots (n+1)_{k-1}}$, where, as usual $(x)_m := x(x+1)\cdots(x+m-1)$) times

$$\frac{(n\,k)!}{n!^k} \quad .$$

Now also fix a specific (*numeric*) cell, $c = [i, j]$, and a specific (numeric) integer r. We want an **explicit** formula, as a **rational function** of n, for the probability that, when you draw (say using the GNW algorithm) *uniformly at random*, one of the $\frac{(n\,k)!(k-1)!}{n!\cdots(n+k-1)!}$ standard Young tableaux, of shape n^k, that the occupant of the cell $c = [i, j]$ is the integer r, in symbols:

$Pr(T_{ij} = r)$.

We will soon explain how to do it, but you are welcome to try it out first using our Maple package `SYT.txt`. Let's give a few examples.

• To get the explicit expression for the probability that the occupant of cell $[1, 3]$ in a random standard Young tableau of shape (n, n, n) happens to be 7, type:

`OcCs([n,n,n],7,[1,3]);` ,

getting

$$\frac{5n\,(n+1)^2\,(n+2)}{9\,(3n-1)\,(3n-2)\,(3n-4)\,(3n-5)} \quad .$$

• For a more complicated example, to get the expression for the probability that the occupant of cell $[3, 3]$ in a random standard Young tableau of shape (n, n, n) happens to be 13, type:

`OcCs([n,n,n],13,[3,3]);`

getting

$$\frac{110n^2\,(-1+n)\,(n+1)^2\,(2+n)\left(233n^2-1933n+3984\right)}{81\,(-1+3n)\,(-2+3n)\,(-4+3n)\,(-5+3n)\,(-7+3n)\,(-8+3n)\,(-10+3n)\,(-11+3n)} \quad .$$

• For yet another example, regarding the three-rowed shape (n_1, n_2, n_3), to get the rational function (in n_1, n_2, n_3) for the probability that cell $[1, 2]$ would be occupied by 3, type

`OcCs([n[1],n[2],n[3]],3,[1,2]);` ,

getting

$$\frac{n_1^2n_2+n_1^2n_3+n_1n_2^2+2n_1n_2n_3+n_1n_3^2+n_2^2n_3+n_2n_3^2-n_1n_2-n_1n_3+n_2^2-n_3n_2+2n_3^2-2n_2-6n_3}{(n_1-2+n_2+n_3)\,(n_1-1+n_2+n_3)\,(n_1+n_2+n_3)} \quad .$$

• If the cell is at the first row, $c = [1, j]$, for some $j > 1$, then there are only finitely many possible occupants r, namely $r = j, j+1, \ldots, k(j-1)+1$, and to get the probability generating function, using the variable x, type

`OcGFs1(L,j,x);` .

For example, entering: `OcGFs1([n,n,n],2,x);`

gives you

$$\frac{2\,(-1+n)\,x^2}{-1+3n}+\frac{8\,(-1+n)\,(n+1)\,x^3}{3\,(-1+3n)\,(-2+3n)}+\frac{(n+1)\,(2+n)\,x^4}{3\,(-1+3n)\,(-2+3n)}\quad ,$$

meaning that the cell $[1,2]$ in a standard Young tableau of shape (n,n,n) is occupied by either 2, 3, or 4, with respective probabilities of $\frac{2(-1+n)}{-1+3n}$, $\frac{8(-1+n)(n+1)}{3(-1+3n)(-2+3n)}$, and $\frac{(n+1)(2+n)}{3(-1+3n)(-2+3n)}$.

To get the limiting distribution as $n\rightarrow\infty$, as well as the expectation, variance, and the first few moments up to the K-th, try `OcGFs1L(L,n,i,x,K);` .

How does Maple perform these amazing calculations? In other words how does it compute $Pr(T_{i,j}=r)$ for a random standard Young tableau of a symbolic shape?

Given a (symbolic, or numeric) shape λ, a cell $c=[i,j]$, and an integer r, how can it happen that $T_{ij}=r$? The cells occupied by $\{1,2,\dots,r\}$ form a certain standard Young tableau with r cells, that is a certain subshape, that must contain the cell $c=[i,j]$, that must be a corner, of course. So let's ask our beloved computer to find all the shapes with r cells that contain the cell $[i,j]$ as a corner, or equivalently the set of partitions, ν, of r with at least i rows such that $\nu_i=j$. Let's call this (finite) set $S([i,j],r)$.

Let, as usual, $f^{\lambda/\nu}$ denote the number of standard Young tableau of **skew-shape** λ/ν (recall that these are shapes where ν is a subshape of λ, and the cells of ν are removed). Then our quantity of interest is

$$\sum_{\nu\in S([i,j];r)} f^{\nu'}f^{\lambda/\nu}\quad ,$$

where ν' is the shape ν with the cell $[i,j]$ removed.

The **number** $f^{\nu'}$ is easily computed using the Young-Frobenius formula. How do we compute the (symbolic) **expression** $f^{\lambda/\nu}$?

Recall that standard Young tableaux of shape $\lambda=(\lambda_1,\dots,\lambda_k)$ are in bijection with **walks** from the **origin** to the point λ in the k-dimensional discrete lattice $\mathcal{N}^k$, that always must stay in the region

$$x_1\geq x_2\geq\ldots\geq x_k\geq 0\quad .$$

Similarly, standard Young tableaux of *skew-shape* λ/ν are in bijection with such 'sub-diagonal' walks from ν to λ. Now following the ideas of André [Z] (see also [GeZ]), put mirrors on the hyperplanes

$$x_1-x_2=-1\quad,\quad x_2-x_3=-1\quad,\quad\ldots\quad,\quad x_{k-1}-x_k=-1\quad,$$

and look at the set of $k!$ images of the point ν under the action of the group generated by these $k-1$ reflections. As is well-known (and fairly easy to see), the underlying group is the symmetric group S_k, and the sign is 1 or -1 according to whether the number of inversions is even or odd. Calling the set of images $IMAGE(\nu)$, we have:

$$f^{\lambda/\nu} = \sum_{\mu \in IMAGE(\nu)} \pm W(\mu, \lambda) \quad ,$$

where $W(\mu, \lambda)$ is the **number** of walks in the lattice from μ to λ given by the **multinomial coefficient**

$$\frac{(\lambda_1 + \ldots + \lambda_k - \mu_1 - \ldots - \mu_k)!}{(\lambda_1 - \mu_1)!(\lambda_2 - \mu_2)! \cdots (\lambda_k - \mu_k)!} \quad .$$

But since we are interested in probabilities, we can divide everything by f^λ and stay in the realm of rational functions.

This is implemented in procedures `Swee(L,M)`.

Computing the Sorting Probabilities for symbolic shape and any two cells where one of them is at the first row

The **numeric** procedure `Pr(L,c1,c2)`, for a random standard Young tableau of shape `L`, manually finds the sorting probability of cell `c1` vs. cell `c2`, and the numeric procedure `MinPr(L)` finds the minimal sorting probabilities among all pair of cells, followed by the 'champions'. For example, if you type

`MinPr([10,4,3]);` ,

you would get

`1/273, {[[1, 5], [3, 1]]}`,

meaning that the minimum (absolute value) of the sorting probabilities among all the $\binom{17}{2} = 136$ pairs of cells is $\frac{1}{127}$ and it is achieved with the pair of cells $[1, 5]$ and $[3, 1]$. But we want to do things **symbolically**. Alas, things get complicated if neither cells are at the first row.

But we can, exactly, and **symbolically**, compute a closed-form expression, as a rational function of the symbols, of the sorting probabilities between any cell $[1, j]$ on the first row and any cell below it (to the left, of course, or else the sorting probability is trivially -1).

This is implemented in procedure `PrS(L,j,c2)`, where `L` is the symbolic shape and `c2` is the cell below the first row that we compare it to `[1,j]`. For example, to get the sorting probability of cell $[1, 3]$ vs. the cell $[2, 2]$, for the shape (n, n, n), type

`PrS([n,n,n],3,[2,2]);` getting

$$-\frac{(17n - 4)(n - 3)}{3(3n - 1)(3n - 4)} \quad .$$

How Does Maple find The Symbolic Sorting Probabilities?

How do we do it? Look at all the possible occupants of cell $c_1 = [1, j]$ (there are finitely many of them). Suppose it happens to be r. How can it be larger than the occupant of cell $c_2 = [m_1, m_2]$?. We find the (finite) set of shapes with r cells that include $c_1 = [1, j]$, and in addition it is a corner. In other words all the shapes ν with r cells such that $\nu_1 = j$, ν has at least m_1 rows, and $m_2 \leq \nu_{m_1}$.

As before add-up $f^{\nu'}$ times $f^{\lambda/\nu}$, and then add-them-up for all possible legal occupants of $[1, j]$. Getting a nice (or not so nice, but sill explicit) expression for $Pr(T_{1,j} > T_{m_1,m_2})$, and hence for the sorting probability $2\,Pr(T_{1,j} > T_{m_1,m_2}) - 1$.

Of course, we always divide by f^λ (but this is already built-in in all our *macros*).

A one-line proof that the Minimal sorting probabilities for the Catalan Poset is $O(\frac{1}{n})$

In a deep and beautiful work [CPP2], the authors proved that the minimal sorting probability of the Young lattice, as the shapes get larger, tends to 0. In the more specific paper [CPP1], they proved, by an ingenious and delicate asymptotic analysis, that for the two-rowed case, $[n, n]$, (what they call the *Catalan poset*), it is $O(\frac{1}{n^{\frac{5}{4}}})$. But using our Maple package, we can get, *without human effort*, a (rigorous!) proof that it is at least $O(\frac{1}{n})$.

Indeed, entering in our Maple package `SYT.txt`, the command :

`PrS([n,n],3,[2,1]);` ,

we get, in one *nano-second* :

$$\frac{3}{2n-1} \quad .$$

So we have the following computer-generated proposition (that admittedly could have been done by humans only using paper and pencil).

Proposition: The sorting probability of the cell $[1, 3]$ and the cell $[2, 1]$ in a random standard Young tableau of shape (n, n) is

$$\frac{3}{2n-1} = \frac{3}{2} \cdot \frac{1}{n} + \frac{3}{4} \cdot \frac{1}{n^2} + \frac{3}{8} \cdot \frac{1}{n^3} + O(\frac{1}{n^4}) \quad .$$

Hence the minimal sorting probability of the Catalan lattice is $O(\frac{1}{n})$.

Similarly, typing

`PrS([n,n],5,[2,2]);`

gives the following proposition.

Proposition: The sorting probability of the cell $[1,5]$ and the cell $[2,2]$ in a random standard Young tableau of shape (n,n) is

$$\frac{45n^2 - 135n + 30}{2\,(2n-5)\,(2n-1)\,(2n-3)} = \frac{45}{16} \cdot \frac{1}{n} + \frac{135}{32} \cdot \frac{1}{n^2} + \frac{75}{16} \cdot \frac{1}{n^3} + O(\frac{1}{n^4})$$

Hence, again, the minimal sorting probability of the Catalan poset is $O(\frac{1}{n})$.

Procedure `FindZero(L,n,K)` searches for all pairs of cells $c_1 = [1,j], c_2 = [m_1, m_2]$ where c_1 is in the first row, and $j, m_2 \leq K$, such that the sorting probability tends to 0 (and hence is, of course $O(1/n)$). Alas, except for the above two pairs (for the Catalan poset), none exists for $K = 100$. Note that here we really lucked out, since the pairs $\{[1,3],[2,1]\}$ and $\{[1,5],[2,2]\}$ are *numeric* (and small), and give upper bound for the minimal sorting probability. In order to get to the *true* minimum, **both** c_1 and c_2 must be *symbolic* (that what was essentially done in [CPP1] and [CPP2] with great human effort).

The special case of the Catalan poset (2-rowed tableaux)

For the Catalan case things can get much faster (as noticed in [CPP1]) and the procedures implementing this can be found by typing `ezraD();` .

`Anij(n,i,j)` is a faster version of PrS([n,n],i,[2,j]). It turns out that in this case we can get closed-form expressions, for the occupancy distribution of an arbitrary cell $[1,i]$ at the first row of a standard Young tableau of shape (n,n) for **symbolic** i, that entail, in turn, closed-form expressions for the *limiting distribution* as n goes to ∞, and **surprise!** we can get explicit expressions for the average, variance, and higher moments for that limiting distribution for symbolic i, and even the *meta-limiting* behavior, as i goes to infinity.

We have

Proposition: The expectation of the occupant of cell $[1,i]$ in a random standard Young tableau of shape (n,n), as n goes to infinity is

$$2i + 2 - \frac{2 \cdot 4^{-i}\,(1+2i)!}{i!^2} \quad ,$$

confirming Richard Stanley's observation mentioned in [CPP1], Eq. (5.1). The asymptotics is

$$2i + 2 - \frac{4}{\sqrt{\pi}} i^{1/2} - \frac{3}{2\sqrt{\pi}} i^{-1/2} + \frac{7}{32\sqrt{\pi}} i^{-3/2} - \frac{9}{256\sqrt{\pi}} i^{-5/2} + O(i^{-7/2})$$

The variance is

$$-\frac{4 \cdot 16^{-i}\,(1+2i)!^2}{i!^4} - \frac{2 \cdot 4^{-i}\,(1+2i)!}{i!^2} + 6i + 6 \quad .$$

The limiting (as i goes to infinity) *skewness* is $\frac{2(5\pi-16)\sqrt{2}}{(3\pi-8)^{\frac{3}{2}}} = -0.4856928234\ldots$

The limiting (as i goes to infinity) *kurtosis* is $\frac{15\pi^2+16\pi-192}{(3\pi-8)^2} = 3.108163850\ldots$

The limiting (as i goes to infinity) *scaled-fifth-moment* is $\frac{2\left(51\pi^2-80\pi-256\right)\sqrt{2}}{(3\pi-8)^{\frac{5}{2}}} = -4.642979574\ldots$

The limiting (as i goes to infinity) *scaled-sixth-moment* is $\frac{105\pi^3+648\pi^2-2240\pi-2560}{(3\pi-8)^3} = 18.66866547\ldots$

For more details see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT9new.txt` .

We note that, by miracle, the (limiting, as n goes to infinity) average, variance, and *any* higher moment, happened to be *gosperable* so Maple is able to evaluate them in closed-form using the Maple command `sum`. We doubt whether this will happen for more rows, but we did not try.

We believe that the Maple package `SYT.txt` can be used to explore further and possibly suggest improvements to the already very impressive work in [CPP1] and [CPP2].

Sample Output

The web-page of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/syt.html` ,

contains lots of output file. Let's just mention some highlights.

• If you want to see a computer-generated article with lots of explicit expressions (as rational functions of n) for the probability distribution of the occupant of cell $[1, i]$ in a (uniformly-at) random-generated Young tableau of rectangular shape with 2 rows and n columns (i.e. of shape $[n, n]$) for all i between 2 and 40, see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT1.txt` .

If you want to see an abbreviated version, with only the *limiting distribution* as n goes to infinity, but all the way to $i = 60$, as well as the expectation, variance, skewness, and kurtosis, see

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT1L.txt` .

If you want to see the plot of the occupancy distribution of the cell $[1, 40]$ as n goes to ∞, look here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTpic2.html` .

• For the analogous output for 3-rowed rectangular shapes, (i.e. (n, n, n)), see, respectively

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT2.txt` , https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYT2L.txt ,

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTpic3.html .

For the analogous information for up to 8 rows (but with less data) see the above-mentioned front of this article.

• For testing the amazing Greene-Nijenhuis-Wilf algorithm vs. the exact results, see the output file

https://sites.math.rutgers.edu/~zeilberg/tokhniot/oSYTsi1.txt .

To get lots of explicit expressions for sorting probabilities refer to the above web-page.

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https://sites.math.rutgers.edu/~zeilberg/mamarimY/Andre1983.pdf .
[Christian Krattenthaler noticed a very long time ago that on the second page (p. 326), last word on line 3: first → last] .

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