

**Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms
and the
Sharpening of Wilf-Zeilberger Theory**

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Sharp Upper Bounds for the Almkvist-Zeilberger Algorithm

This section is a discrete-continuous analog of [MZ].

Theorem Let

$$F(n, x) = POL(n, x) \cdot H(n, x) \quad , \quad (DiscreteContHyperGeometric)$$

where $POL(n, x)$ is a polynomial of (n, x) , and

$$H(n, x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^P S_p(x)^{\alpha_p} \right) \cdot \left(\frac{s(x)}{t(x)} \right)^n \quad (PureDiscreteContHyperGeometric)$$

where $a(x), b(x), s(x), t(x)$ and $S_p(x)$ ($1 \leq p \leq P$) are *polynomials* of x , while the α_p 's are commuting *indeterminates*, Let

$$L = deg(b) + deg(s) + deg(t) + \sum_{p=1}^P deg(S_p) + max(deg(a), deg(b)) - 1 \quad ,$$

then there exist $L + 1$ polynomials in n , $e_0(n), e_1(n), \dots, e_L(n)$, *not all zero*, and a rational function $R(x, n)$ such that $G(n, x) := R(n, x)F(n, x)$ satisfies

$$\sum_{i=0}^L e_i(n)F(n+i, x) = D_x G(x, n) \quad . \quad (GertDoron)$$

If $F(n, \pm\infty) = 0$ (and hence $G(n, \pm\infty) = 0$) it follows, by integrating from $-\infty$ to ∞ that

$$a(n) := \int_{-\infty}^{\infty} F(n, x) dx \quad ,$$

satisfies the linear recurrence equation with polynomial coefficients

$$\sum_{i=0}^L e_i(n)a(n+i) = 0 \quad .$$

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Proof: Let L , for now, be *any* positive integer. Let

$$\overline{H}(n, x) = e^{a(x)/b(x)} \cdot \left(\prod_{p=1}^P S_p(x)^{\alpha_p} \right) \cdot \frac{s(x)^n}{t(x)^{n+L}} \quad .$$

We have

$$\sum_{i=0}^L e_i(n) F(x, n+i) = h(x) \cdot \overline{H}(x, n) \quad ,$$

where

$$h(x) := \sum_{i=0}^L e_i(n) POL(n+i, x) s(x)^i t(x)^{n-i} \quad .$$

Let $q(x)$ and $r(x)$ be the numerator and denominator, respectively, of the logarithmic derivative of $\overline{H}(n, x)$, i.e.

$$\frac{D_x \overline{H}(n, x)}{\overline{H}(n, x)} = \frac{q(x)}{r(x)}$$

Write

$$G(n, x) = \overline{H}(n, x) \cdot r(x) \cdot X(x) \quad , \quad (\textit{Ansatz})$$

where $X(x)$ is a polynomial to be determined. Now (*GertDoron*) is equivalent to

$$(r'(x) + q(x)) \cdot X(x) + r(x)X'(x) = h(x) \quad (\textit{ContGosper})$$

Let $M := \deg(h) - \max(\deg(r' + q), \deg(r) + 1)$, and write $X(x)$ with undetermined coefficients. Plugging this into (*ContGosper*), and equating coefficients, results in $\deg(h) + 1$ equations for $L + M + 2$ unknowns. In order to guarantee a solution, we need

$$L + M + 2 > \deg(h) + 1 \quad ,$$

in other words

$$(L + M + 2) - (\deg(h) + 2) \geq 0 \quad ,$$

in other words,

$$L \geq \max(\deg(r' + q), \deg(r) + 1) \quad .$$

We leave it to the reader to verify that the expression on the right is

$$L = \deg(b) + \deg(s) + \deg(t) + \sum_{p=1}^P \deg(S_p) + \max(\deg(a), \deg(b)) - 1 \quad \square$$

REFERENCES