Dr. Z's March 10 Colloquium Handout

Problem Type: A function is given by some rule. First conjecture, and then rigorously prove, an explicit formula for it.

Example Problem: Let $f_L(n)$ be the expected life of a gambler in a fair casino, who currently has n dollars and loses or wins a dollar with probability $1/2$, and has to quit when he is ruined, or when he has L dollars. Find an explicit formula for $f_L(n)$ as a function of n and L.

Steps	**Example**
1. Using human (or computer) ingenuity, set up an algorithm for computing the function numerically for "small values" of the arguments.	**1.** For a fixed L, the $n+1$ quantities $f_L(n)$ $(n = 0 \dots L)$ are completely characterized by the system of $n+1$ linear equations with $n+1$ unknowns $$f_L(n) = \frac{1}{2} f_L(n-1) + \frac{1}{2} f_L(n+1) + 1 \quad , \quad (1 \leq n \leq L-1)$$ $$f_0(0) = 0 \quad , \quad f_L(L) = L \quad .$$
2. Using a linear solver, crank out enough data.	**2.** $$f_3(0) = 0, f_3(1) = 2, f_3(2) = 2, f_3(3) = 0 \quad ,$$ $$f_4(0) = 0, f_4(1) = 3, f_4(2) = 4, f_4(3) = 3, f_4(4) = 0 \quad ,$$ $$f_5(0) = 0, f_5(1) = 4, f_5(2) = 6, f_5(3) = 6, f_5(4) = 4, f_5(5) = 0 \quad .$$

3. Conjecture an *ansatz* and use *undetermined coefficients* to find it (if possible).

3. Setting $f_L(n) = an^2 + bn^2 + cn$ and solving for a, b, c (for each specific L) we get

$$f_3(n) = n(3-n) \quad f_4(n) = n(4-n) \quad f_5(n) = n(5-n)$$

$$f_6(n) = n(6-n) \quad , \quad f_7(n) = n(7-n) \quad .$$

4. Using the same methodology conjecture an expression in L for $f_L(n)$, then prove it *rigorously* by using the *definining algorithm*, but now **symbolically**.

4.

$$f_L(n) = n(L-n) \quad .$$

Now let's prove it! Let's denote $g_L(n) := n(L-n)$. We have

$$g_L(n) - \frac{1}{2} g_L(n-1) - \frac{1}{2} g_L(n+1) - 1 =$$

$$L(n-L) - \frac{1}{2}(n-1)(L-(n-1)) - \frac{1}{2}(n+1)(L-(n+1)) - 1 = 0 \quad .$$

Also $g_L(0) = 0 \cdot (L-0) = 0$ and $g_L(L) = L(L-L) = 0$. So we have a **rigorous** proof that $f_L(n) = g_L(n)$ for *every* n and L.