

We prove the identity

$${}_2F_1(-2n, b; -2n+2r-b; -1) = \frac{(\frac{1}{2})_n(b+1-r)_n}{(\frac{1}{2}b + \frac{1}{2} - r)_n(\frac{1}{2}b + 1 - r)_n} {}_3F_2(-n, r, 1-r; \frac{1}{2}, b+1-r; 1)$$

for $r \geq 1$. Multiplying both sides by $(1+b-2r)_{2n}/(2n)!$ gives the equivalent identity

$$\sum_{k=0}^{2n} (-1)^k \binom{b+k-1}{k} \binom{2n+b-2r-k}{2n-k} = \sum_{i=0}^{r-1} 2^{2i} \binom{r+i-1}{2i} \binom{b+n-r}{n-i}. \quad (1)$$

Let

$$L_{m,r} = \sum_{k=0}^m (-1)^k \binom{b+k-1}{k} \binom{m+b-2r-k}{m-k}$$

and let

$$R_{n,r} = \sum_{i=0}^{r-1} 2^{2i} \binom{r+i-1}{2i} \binom{b+n-r}{n-i}.$$

We will show that

$$\sum_{n=0}^{\infty} \sum_{r=1}^{\infty} L_{2n,r} x^{2n} y^r = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} R_{n,r} x^{2n} y^r. \quad (2)$$

It is easy to compute

$$\sum_{m=0}^{\infty} L_{m,r} x^m = (1-x)^{2r-b-1} (1+x)^{-b}.$$

Multiplying by y^r and summing on r from 1 to infinity gives

$$T(x, y) := \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} L_{m,r} x^m y^r = y \frac{(1-x)^{1-b} (1+x)^{-b}}{1-y(1-x)^2}.$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} L_{2n,r} x^{2n} y^r &= \frac{1}{2} (T(x, y) + T(-x, y)) \\ &= \frac{y(1-y+yx^2)}{(1-y(1-x)^2)(1-y(1+x)^2)(1-x^2)^b}. \end{aligned} \quad (3)$$

A straightforward computation shows that the right side of (2) is also equal to (3).