TOWARDS A WZ EVOLUTION OF THE MEHTA INTEGRAL*

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Abstract. The celebrated Mehta integral is shown to be equivalent to a simple algebraic-differential identity, which is completely routine for any fixed number of variables.

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Askey [As] proposed the problem of proving the Mehta (see [M]) integral identity

$$\text{(Mehta)} \quad \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-x_1^2/2 - \cdots - x_n^2/2) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2c} dx_1 \cdots dx_n = \prod_{j=1}^n \frac{(cj)!}{j!}$$

without using Selberg's integral (see [M]). This problem was solved by Anderson [An]. Here we use the method of [WZ1] and [WZ2] to initiate another Selberg-free proof that we believe is of independent interest. We show that (Mehta) for any given n is equivalent to the following elegant identity ($d := n(n-1)/2$):

$$\begin{aligned} \text{(Mehta')} \quad & \left\{ \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \right\}^d \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = 2^d d! n! \\ & \cdot \prod_{1 \leq s < r \leq n} (rc + s). \end{aligned}$$

(Mehta') is purely routine for any specific n , but at the time of writing we are unable to prove it directly for general n . Of course, we do have a proof, since we are going to show that (Mehta') and (Mehta) are equivalent, but what we are after is a *direct* proof. The author is offering a prize of 25 US dollars for such a proof.

The present method also obviously extends to the Macdonald–Mehta integral [M], which was proved by Beckner and Regev [BR] for the classical root systems (see [M]), by Garvan [G] for the exceptional root system F_4 , and by Opdam [O] for E_6 , E_7 , and E_8 . It follows that the present approach should also yield new proofs for all the exceptional root systems, at least in principle, but most likely in practice also. More important, it seems to have a high chance of producing a uniform, intrinsic, classification-independent proof. We leave to the reader, as an instructive exercise, the task of finding the root-system analog of (Mehta') that is equivalent to the now-proved Mehta–Macdonald conjecture, and we offer an additional 25 dollars for an intrinsic proof.

Our proposed proof of (Mehta) will be a *derivation* rather than a *verification*, and will follow the method of [WZ2]. Let us call the left of (Mehta) $L(c)$, and the integrand $F(c; x_1, \dots, x_n)$. We know from the general theory of [WZ2] that for some r

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and some rational functions $P_1, \dots, P_n$, in $(c, x_1, \dots, x_n)$, and some rational functions in $c, a_0(c), \dots, a_r(c)$,

$$(WZ) \quad \sum_{s=0}^r a_s(c) F(c+s; x_1, \dots, x_n) = \sum_{i=1}^n \frac{\partial}{\partial x_i} (P_i F).$$

Let us be optimistic and try out $r = 1$. Without loss of generality, set $a_1 := 1$. Substituting F in (WZ), performing all differentiations, and dividing throughout by F leads to the following equation for P_i and a_0 :

$$(1) \quad a_0 + \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = \sum_{i=1}^n \frac{\partial P_i}{\partial x_i} + 2c \sum_{i=1}^n \left(\sum_{j \neq i} \frac{1}{x_i - x_j} \right) P_i - \sum_{i=1}^n x_i P_i.$$

To be even more optimistic, assume that P_i are polynomials, rather than mere rational functions, in their dependence on $(x_1, \dots, x_n)$, and are furthermore the components of the gradient of another polynomial P , i.e., $P_i = \partial P / \partial x_i$, $i = 1, \dots, n$, for some polynomial P . Equation (1) then becomes

$$(2) \quad a_0 + \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = \left\{ \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i=1}^n \left(\sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} \right\} P - \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} \right) P.$$

If we can find such a polynomial P , and compute the corresponding a_0 , then it would follow from (WZ), upon integrating with respect to $x_1, \dots, x_n$, that $L(c)$ satisfies the recurrence $L(c+1) = -a_0(c)L(c)$, which combined with $L(0) = 1$ would enable one to find $L(c)$. Note that the mere existence of a_0 , which we will shortly prove, is given by the left side of (Mehta') and implies that $L(c)$ is of *closed form*, which from a theoretical point of view is almost as good as knowing what it is exactly.

Let us write P as a sum of its homogeneous parts

$$P = \sum_{j=2}^{2d} P^{(j)}, \quad P^{(j)} \text{ homog. of deg. } j,$$

where, as above, d equals $n(n-1)/2$. Denote the operator inside the braces of (2) or (Mehta') by $\mathbf{Z}$. Using Euler's formula, we get

$$(3) \quad a_0 + \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 = \sum_{j=2}^{2d} \mathbf{Z} P^{(j)} - \sum_{j=2}^{2d} j P^{(j)} = (-2d) P^{(2d)} + \sum_{j=2}^{2d} (\mathbf{Z} P^{(j)} - (j-2) P^{(j-2)}).$$

By equating corresponding homogeneous parts, we get

$$(4) \quad P^{(2d)} = -(2d)^{-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2, \quad P^{(j-2)} = \frac{1}{j-2} \mathbf{Z} P^{(j)}, \quad j = 2d, 2d-2, \dots, 4.$$

It is easy to see that $\mathbf{Z}$ maps homogeneous symmetric polynomials to homogeneous symmetric polynomials, and that it reduces the degree by 2. Iterating (4) and comparing the constant part of (3) finally yields that both P and a_0 indeed exist, and that

$$a_0 = -(2^d d!)^{-1} \mathbf{Z}^d \left[\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \right].$$

Hence Mehta's integral is indeed expressible in closed form, and proving that its value coincides with the value implied by (Mehta) amounts to proving (Mehta').

The referee empirically noticed the following generalization. For a , any positive integer, we have

$$\begin{aligned} \text{(Mehta'')} \quad & \left\{ \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq n} \frac{1}{x_i - x_j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \right\}^{ad} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2a} \\ & = 2^{ad} (da)! n! a \prod_{2 \leq r \leq n} \prod_{1 \leq s \leq ra, s \not\equiv 0 \pmod r} (rc + s). \end{aligned}$$

It turns out that (Mehta'') follows from (Mehta) exactly the same way as (Mehta'); just replace c by ca , in (Mehta), and repeat the argument! In particular, it follows that (Mehta') implies (Mehta''), albeit indirectly, via (Mehta).

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