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Deconstructing the Zeilberger algorithm†

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By looking under the hood of Zeilberger's algorithm, as simplified by Mohammed and Zeilberger, it is shown that all the classical hypergeometric closed-form evaluations can be discovered *ab initio*, as well as many "strange" ones of Gosper, Maier, and Gessel and Stanton. The accompanying Maple package `FindHypergeometric` explains the various miracles that account for the classical evaluations, and the more specialized Maple package `twoFone`, also accompanying this article, finds many "strange" ${}_2F_1$ evaluations, and these discoveries are in some sense, exhaustive. Hence WZ theory is transgressing the boundaries of the *context of justification* into the *context of discovery*.

Keywords: WZ theory; Zeilberger algorithm; Neo-classical approach; Miracle

NOTATION *For k integer,*

$$(z)_k := z(z+1)\dots(z+k-1), \quad \text{if } k \geq 0$$
$$(z)_k := \frac{1}{(z+k)_{-k}} \qquad \text{if } k < 0.$$

Prerequisites: We assume familiarity with [10].

"That's very Nice that you Computers can Prove Identities, But you Still Need Us Humans to Conjecture Them!", Well, No Longer!

Recall that the simplified Zeilberger algorithm [10] inputs a *proper hypergeometric* term:

$$F(n,k) = \mathrm{POL}(n,k)\cdot H(n,k), \tag{1}$$

where $\mathrm{POL}(n,k)$ is a polynomial in (n,k) and

$$H(n,k) = \frac{\prod_{j=1}^{A}\left(a_j''\right)_{a_j'n+a_jk}\prod_{j=1}^{B}\left(b_j''\right)_{b_j'n-b_jk}}{\prod_{j=1}^{C}\left(c_j''\right)_{c_j'n+c_jk}\prod_{j=1}^{D}\left(d_j''\right)_{d_j'n-d_jk}}z^k, \tag{2}$$

where a_j, a_j', b_j, b_j', c_j, c_j', d_j, d_j' are *non-negative integers*, and z, a_j'', b_j'', c_j'', d_j'' are *commuting indeterminates*. It gives a non-negative integer L, polynomials (of n) $e_0(n), e_1(n), \dots, e_L(n)$

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and a rational function (of n and k) $R(n,k)$, such that, if $G(n,k) := R(n,k)F(n,k)$, then

$$\sum_{i=0}^{L} e_i(n)F(n+i,k) = G(n,k+1) - G(n,k). \tag{3}$$

Assuming that $F(n,\pm\infty) = 0$ (as is often the case), we have, by adding w.r.t. k, and noting that the sum on the right telescopes is 0, that

$$a(n) := \sum_{k=-\infty}^{\infty} F(n,k)$$

satisfies a homogeneous linear *difference* equation with polynomial coefficients:

$$\sum_{i=0}^{L} e_i(n)a(n+i) = 0. \tag{4}$$

If the *order*, L, happens to be 1, then the recurrence can be solved in *closed-form*, and we have a closed-form evaluation.

In particular, the Zeilberger algorithm (as implemented in my own Maple package `EKHAD`, and starting with Maple 6, in the built-in package `SumTools[Hypergeometric]`, and there is a very popular Mathematica implementation by Paule and Schorn [11] can immediately discover the right hand side, if the left hand side is given. For example, if one inputs

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^3,$$

one would get back a first-order recurrence for the sum, that immediately entails the closed-form evaluation $(3n)!/n!^3$.

Since the set of conceivable hypergeometric summands (that humans or computers can write down) is countable, one can arrange them in lexicographic order, and eventually, just like in Hilbert's dream, get to any specific hypergeometric sum, and get the recurrence it satisfies (and with Petkovsek's [12] celebrated algorithm, see also [13], one can guarantee that it is minimal). If we are only interested in finding *closed-form* evaluations, i.e. the cases when $L = 1$, then we can just discard all the cases when we get $L > 1$, and then publish a book of "closed-form evaluations" of any bounded "complexity".

But, in this way it may take a thousand years to get to Saalschütz or Dixon, and a million years to get to Dougall. This article outlines an *efficient* algorithm for outputting *all* hypergeometric closed-form evaluations of any bounded "complexity". Unfortunately, if this complexity gets higher, it runs out of time and memory, but it can outperform by orders of magnitude an exhaustive search.

Already, WZ theory has a mechanism for generating new identities by the process of *specializing and dualizing* [13,14]. This technique was elevated to an art form by Ira Gessel [5]. However, if we do this randomly, we would get lots of new such evaluations, but the summands, while technically proper-hypergeometric, are usually extremely messy, in the sense that they are far from being *purely-hypergeometric*, i.e. their *polynomial part* ($\mathrm{POL}(n,k)$ above) is of high degree. Also, this approach needs the classical identities (Saalschütz, Dixon, Dougall etc.) as *starters*.

In the present approach we can rediscover from scratch, in a natural way, all the classical, and the so-called *strange* [6] hypergeometric closed-form evaluations. The algorithm also has the potential to discover many new such strange identities.

1. Robert Maier's neo-classical approach

Hypergeometric series started out as *solutions* of certain ordinary differential equations. Using this fact, Euler, Gauss, Kummer, Riemann, Goursat and other giants, found *transformation formulas*, that lead to the classical evaluations. On the other hand, WZ theory considers the erstwhile parameters as *active*, *discrete* variables.

Recently, Robert Maier [8] found a brand-new transformation formula for (a general!) ${}_{r+1}F_r$, that obeys *algebraic constraints*. It is not proved, but later discovered with WZ theory. So the moral is: *find new approaches but keep the old ones ...*.

2. Under the hood of Zeilberger

Abramov and Le [1,2] showed that the (original) Zeilberger algorithm sometimes works even for hypergeometric summands that are *non-proper*. However, for the most important case of *proper*-hypergeometric summation, the Zeilberger algorithm has been considerably simplified in [10], where it is shown that there is a *sharp* upper bound for L, which is really what it should be, if $F(n,k)$ is replaced by $F(n,k)x^k$. But in special cases, L may be smaller. Experiment with procedure `DoronMiracles` in the Maple package `FindHyperGeometric` accompanying this article is strongly recommended. `DoronMiracles` is a *verbose rendition*, that traces the algorithm, and lists any "miracles" that happen which reduce L from its generic promised value.

Recall from [10] that everything boils down to solving the linear equation

$$f(k)X(k+1) - g(k-1)X(k) - h(k) = 0, \tag{5}$$

where $f(k)$, $g(k)$ and $h(k)$ are certain polynomials derivable from the input; $h(k)$ depends linearly on the unknowns $e_i(n)$s; and the coefficients of $X(k)$ are also unknowns. The argument in [10] displays an L, and a degree, M, for $X(k)$, such that if one substitutes a generic polynomial of degree M in k, for $X(k)$, expands equation (5), and then sets all the coefficients of the resulting polynomial in k to 0, one gets a system of *homogeneous* linear equations with *more unknowns than equations*, and hence with a *guarantee* for a non-zero solution.

3. Miracles

However, sometimes pleasant surprises occur, and the guaranteed L can be made lower, due to *miracles*.

A *miracle of the first kind* happens when there is a potential cancellation in the left of equation (5). This happens when the degrees (in k) and the leading coefficients of $f(k)$ and $g(k)$ are the same. Then the potential degree gets increased by 1.

A *miracle of the second kind* can only happen in the wake of a miracle of the first kind. It is an extremely rare event. It happens when the potential degree of $X(k)$ can be made yet higher. To test, write the leading and second-to-leading coefficients of $X(k)$ in generic form, with *generic* degree, plug into equation (5), and look at the leading coefficient and set it to 0. We will get a certain equation for that assumed degree. Usually (and certainly generically), the solution would be *symbolic*, and hence impossible, but if it happens to be a *numeric* integer, and it exceeds the proposed degree promised by the first miracle, then we do indeed have a miracle of the second kind. Interestingly, Apéry's celebrated sum

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2,$$

whose "generic" order is 4, is actually of order 2, because it is a beneficiary of this extremely rare miracle of the second kind.

Finally, once we settled for the highest-possible-degree for $X(k)$, and replaced it in equation (5) by a generic polynomial of that degree, with undetermined coefficients, we expand everything, and set all the coefficients of the polynomial, in k, of the left side of equation (5), to 0. We get a system of homogeneous linear equations for a certain set of unknowns. These unknowns consist of the coefficients of $X(k)$, as well as the coefficients, $e_i(n)$, of the desired recurrence. Sometimes having the first and/or second miracle is already enough to have more unknowns than equations, but in the contrary case, there is still *hope*.

Indeed, a system of linear equations, with equations more than unknowns *may* have a non-zero solution. All we need is that a certain determinant (or determinants) vanish. In such case, we have a *miracle of the third kind*.

Note that the third miracle may still happen even if the first and second ones did not. Sometimes the first miracle suffices by itself, sometimes we need the first and the second, sometimes we need the first and the third, sometimes we even need *all three miracles* (see below), but many times the third miracle by itself suffices.

4. The miracles that Gauss, Kummer, Saalschütz, Dixon, and Dougall should be grateful for

By running `DoronMiracles` in the Maple package `FindHyperGeometric` (type `ezra(DoronMiracles)`: there, for help), we have the following.

Gauss's ${}_2F_1(a,b;c;1)$ happens because of the *first miracle*, that suffices.

Gauss's ${}_2F_1(2a,2b;a+b+1/2;1/2)$ happens because of the *third miracle*. The first miracle didn't happen, but the *third* one saved the day. This seems to be the case in all the *strange* evaluations, at least for ${}_2F_1$s [4].

Likewise, Kummer's ${}_2F_1(a,b;1+a-b;-1)$ happens only because of the *third miracle*. So it deserves the name *strange*, even though it has two parameters.

The celebrated Pfaff-Saalchütz four-parameter ${}_3F_2(a,b,-n;c,1+a+b-c-n;1)$ evaluation happens because of the *first and second* miracles. The generosity of the second miracle produces less equations than unknowns (in the last, solving equation (5) phase), hence there was no need for another miracle.

For the equally celebrated Dixon three-parameter ${}_3F_2(a,b,c;1+a-b,1+a-c;1)$ evaluation, the *first miracle* did happen. There was *no second miracle*. But even though equation (5) demands a set with three equations and three unknowns to be solved, that has *a priori* probability 0 of success, nevertheless it can be solved, thanks to a *miracle of the third kind*.

Last but not least, the Dougall ${}_7F_6$ giant (which, in our notation, is really a mere ${}_6F_5$ with a linear polynomial in front) must be grateful to *all three miracles*.

To summarize:

Gauss: **1**.
GaussHalf, Kummer, all "strange" ${}_2F_1$: **3**.
Pfaff-Saalscütz: **1**,**2**.
Dixon: **1**,**3**.
Dougall: **1**,**2**,**3**.

5. How to manufacture miracles by tweaking the Zeilberger algorithm

Recall that the input has the form

$$F(n,k) = \mathrm{POL}(n,k) \cdot H(n,k), \tag{1}$$

where $\mathrm{POL}(n,k)$ is a polynomial in (n,k) and

$$H(n,k) = \frac{\prod_{j=1}^{A} \left(a''_j\right)_{a'_j n + a_j k} \prod_{j=1}^{B} \left(b''_j\right)_{b'_j n - b_j k}}{\prod_{j=1}^{C} \left(c''_j\right)_{c'_j n + c_j k} \prod_{j=1}^{D} \left(d''_j\right)_{d'_j n - d_j k}} z^k, \tag{2}$$

where the a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j are *non-negative integers*, and z, a''_j, b''_j, c''_j, d''_j are *commuting indeterminates*.

Now fix the a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j (this cannot be helped, at least for the present approach), but keep the a''_j, b''_j, c''_j, d''_j and z as *indeterminates* and ask for what *specialization* will one or more of the above miracles happen. It is very easy for the computer to find conditions for the first miracle, and also for the second (usually it does not happen). The hardest miracle to perform (computationally) is the third. We have to set a certain determinant (or determinants) to zero, and get, this time, a set of *non-linear* (polynomial) equations for the a''_j, b''_j, c''_j, d''_j's and z. *A priori*, there may be no solution (and indeed often no miracle is possible), but whenever there is a solution, the computer can find it, since it knows, thanks to Bruno Buchberger and his Gröbner bases, how to solve a system of polynomial equations. For now, we are using Maple's built-in implementation, but it may be a good idea to use special-purpose programs like Macaulay, SINGULAR, or MAGMA.

Of course, we are unable to guarantee that we found all hypergeometric identities, even not all ${}_2F_1$s, but the Maple package `twoFone` (to be hopefully followed by packages like `threeFtwo`) finds *all* tuples (a,b,c,b',c',z) such that

$${}_2F_1\left(\begin{matrix} -an, bn + b' \\ cn + c' \end{matrix}; z\right)$$

admits a closed-form evaluation and the integers a,b,c lie in the range $1 \le a \le K_1$, $-K \le b, c \le K$, for any inputted positive integers K_1, K. The computer also discards all specializations of the classical identities of Gauss and Kummer, as well as any consequences of previously discovered identities via the Euler and the two Pfaff transformations (see [3, Ch.2, equations (2.2.6), (2.2.7) and (2.3.14)]). So the final listing contains *mutually independent* genuinely new "strange" identities. It turns out that for the ${}_2F_1$ case, most of them were already known [6,8], yet some of them seem brand-new.

6 Future work

We have only scratched the surface. It would be interesting to generalize the package `twoFone` to `threeFtwo` etc. This is a non-trivial programming task, since we would have to teach the computer to automatically separate the wheat from the chaff, and discard the many identities that are either specializations of classical evaluations, or equivalent to previously-discovered ones, via one of the known (or newly-discovered) transformation formulas.

Another interesting problem is to explain, via the present approach, why the recurrence for the sum of the $2r$th powers of the binomial coefficients has only order r, rather than $2r$. At present we can do it for each and every *numeric* r, and it turns out (for $r \leq 7$), that this happens thanks to the first and third miracles, but it would be nice to prove it *in general*, i.e. for *symbolic* r.

Mohamud Mohammed [9] is currently working on a q-analog, i.e. an analogous treatment for the q-Zeilberger algorithm.

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