1. a) \( \lim_{x \to 1} \frac{\ln x}{x^2 - 1} = \frac{0}{0} \) Indeterminate form

Apply L'Hôpital's: \( \lim_{x \to 1} \frac{\ln x}{x^2 - 1} = \lim_{x \to 1} \frac{1/x}{2x} = \frac{1}{2} \)

b) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) = \infty - \infty \) Indeterminate form

\[ \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \frac{0}{0} \] Indeterminate form

Apply L'Hôpital's: \( \lim_{x \to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \frac{0}{0} \) Indeterminate form

Apply L'Hôpital's again: \( \lim_{x \to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \lim_{x \to 0} \frac{e^x}{e^x + e^x + xe^x} = \frac{1}{2} \)

c) \( \lim_{x \to \infty} x \left( \arctan x - \frac{\pi}{2} \right) = \infty \cdot 0, \) since \( \lim_{x \to \infty} \arctan x = \frac{\pi}{2} \)

\[ \lim_{x \to \infty} x \left( \arctan x - \frac{\pi}{2} \right) = \lim_{x \to \infty} \arctan x - \frac{\pi}{2} = 0 \] Indeterminate form

Apply L'Hôpital's: \( \lim_{x \to \infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} = \lim_{x \to \infty} \frac{1}{1 + x^2} = \lim_{x \to \infty} \frac{-x^2}{1 + x^2} = -\infty \)

Apply L'Hôpital's again: \( \lim_{x \to \infty} \frac{-x^2}{1 + x^2} = \lim_{x \to \infty} \frac{-2x}{2x} = -1 \)

2. a) \( \lim_{x \to \infty} \frac{x^4 - 2x}{e^x + 1} = \frac{\infty}{\infty} \) Indeterminate form

Apply L'Hôpital's: \( \lim_{x \to \infty} \frac{x^4 - 2x}{e^x + 1} = \lim_{x \to \infty} \frac{4x^3 - 2}{e^x} = \infty \)

Apply L'Hôpital's three more times:

\[ \lim_{x \to \infty} \frac{4x^3 - 2}{e^x} = \lim_{x \to \infty} \frac{12x^2}{e^x} = \lim_{x \to \infty} \frac{24x}{e^x} = \lim_{x \to \infty} \frac{24}{e^x} = 0 \]

\[ \lim_{x \to \infty} \frac{x^4 - 2x}{e^x + 1} = \infty \) (so, no horizontal asymptote on the left)

There is only one horizontal asymptote: \( y = 0 \)
Find the horizontal asymptotes of the function.

b) \( \lim_{x \to \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 + 1} \right) = (\infty - \infty) \) Indeterminate form

Multiply by conjugate:

\[
\lim_{x \to \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 + 1} \right) = \lim_{x \to \infty} \frac{x^2 + 3x - x^2 - 1}{\sqrt{x^2 + 3x + \sqrt{x^2 + 1}}}
= \lim_{x \to \infty} \frac{3x - 1}{\sqrt{x^2 + 3x + \sqrt{x^2 + 1}}} = \infty
\]

Factoring:

\[
\lim_{x \to \infty} \frac{x(3-1/x)}{|x| \left( \sqrt{1 + 3/x + \sqrt{1 + 1/x}} \right)} = \lim_{x \to \infty} \frac{x'(3-1/x)}{\left( \sqrt{1 + 3/x + \sqrt{1 + 1/x}} \right)} = \frac{3}{2}
\]

Since \( \lim_{x \to -\infty} \frac{x}{|x|} = -1, \) \( \lim_{x \to \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 + 1} \right) = -\frac{3}{2} \)

The horizontal asymptote are: \( y = \frac{3}{2} \) and \( y = -\frac{3}{2} \).

c) \( \lim_{x \to \infty} (e^x + 2)^{1/x} = 0^0 \) Indeterminate form

Since \( e^{\ln y} = y \), we have \( \lim_{x \to \infty} (e^x + 2)^{1/x} = \lim_{x \to \infty} e^{\ln(e^x + 2)^{1/x}} = \lim_{x \to \infty} e^{\ln(e^{x+1})} = e^\infty \)

Apply L’Hopital’s

\[
\lim_{x \to \infty} e^{\frac{\ln(e^{x+1})}{x}} = \lim_{x \to \infty} e^{\frac{e^{x+1}/(e^{x+1})}{1}} = e^\infty = L'H \lim_{x \to \infty} e^{x} = 1
\]

\( \lim_{x \to \infty} (e^x + 2)^{1/x} = (0 + 2)^0 = 1 \)

The horizontal asymptote are: \( y = 1 \) and \( y = e \).

3. a) Use Linearization to find an estimate for \( \sqrt{20} \).

Let \( f(x) = \sqrt{x} \). Then \( f'(x) = \frac{1}{2\sqrt{x}} \).

We will find the linearization of \( f \) at \( x = 16 \).

Since \( f(16) = 4 \) and \( f'(16) = \frac{1}{8} \), the linearization of \( f \) at 16 is:

\[
L(x) = f(16) + f'(16)(x - 16) = 4 + \frac{1}{8}(x - 16).
\]

Therefore, \( \sqrt{20} = f(20) \approx L(20) = 4 + \frac{1}{8}(20 - 16) = \frac{9}{2} \).

b) Is your answer from part (a) an overestimate or an underestimate? How do we know?

Since \( f''(x) = -\frac{1}{4x^{3/2}} < 0 \), the function is concave down. Therefore, the tangent line lies above the graph. Thus, the answer in part (a) is an overestimate.
4. It is known that $2x + 1 - x^3$ has a solution in the interval $[1, 2]$.

   a) In order to approximate a solution on this interval by Newton’s Method, how must you rewrite the equation?
   
   Answer: $2x + 1 - x^3 = 0$

   b) Write down the first two approximations $x_1$ and $x_2$ to this solution using Newton’s Method with an initial guess of $x_0 = 1$. Simplify your answers as much as possible.

   Let $f(x) = 2x + 1 - x^3$. Then $f'(x) = 2 - 3x^2$, and $f(1) = 2$, $f'(1) = -1$.

   Therefore, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2}{-1} = 3$.

   Since $f(3) = -20$, $f'(3) = -25$, we have $x_2 = 3 - \frac{-20}{-25} = \frac{11}{5}$.

   Answer: $x_1 = 3$ and $x_2 = \frac{11}{5}$.

5. Consider the function $f(x) = \ln \sqrt{x}$ on the interval $[1, e]$.

   a) $f(x) = \ln \sqrt{x}$ is continuous on $[1, e]$.

   Since $f'(x) = \frac{1}{2x}$ exists for all $x$ on $(1, e)$, the function $f$ is differentiable on $(1, e)$.

   b) Since $f(1) = 0$ and $f(e) = 1/2$, we have $\frac{f(e) - f(1)}{e - 1} = f'(c)$ for some $c$ in the interval $(1, 3)$.

   So, $\frac{1}{2(e - 1)} = \frac{1}{2c} \implies c = e - 1$.

6. Find the absolute minimum and absolute maximum values of the function $f(x) = \frac{x^2}{x - 1}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

   
   $f'(x) = \frac{2x^2 - 2x - x^2}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$. Note $f'(x)$ exists for all $x$ in $[-\frac{1}{2}, \frac{1}{2}]$, and $f'(x) = 0$ when $x = 0$ and $x = 2$. Since $x = 2$ is not in $[-\frac{1}{2}, \frac{1}{2}]$, the only critical point is $x = 0$.

   To find the absolute minima and absolute maxima of the function, we need to evaluate the function at the end points of its domain and at the critical point (by EVT).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1/2</td>
<td>-1/6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>-1/2</td>
</tr>
</tbody>
</table>

   Answer: The absolute maximum is 0 and the absolute minimum is $-1/2$. 
7. Consider the function \( f(x) = (\sin^2 x + 2) \cos x \) on the interval \([-\pi/2, \pi/2]\).

a) Find all the critical points of \( f \).

\[
f'(x) = 2 \sin x \cos^2 x - \sin^3 x - 2 \sin x = \sin x(2 \cos^2 x - \sin^2 x - 2)
\]

Since \( \sin^2 x = 1 - \cos^2 x \), we have

\[
f''(x) = \sin x(3 \cos^2 x - 3) = 3 \sin x(\cos x - 1)(\cos x + 1).
\]

On the interval \([-\pi/2, \pi/2]\), \( f'(x) = 0 \) when \( x = 0 \).

**Answer:** The only critical point of \( f(x) \) is \((0, 2)\).

b) Find the intervals where \( f \) is increasing and the intervals where \( f \) is decreasing.

<table>
<thead>
<tr>
<th>( f'(x) )</th>
<th>-( \pi/2 )</th>
<th>0</th>
<th>( \pi/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>↑</td>
<td>↓</td>
<td>↑</td>
</tr>
</tbody>
</table>

**Answer:** The function is increasing on \((-\pi/2, 0)\) and it is decreasing on \((0, \pi/2)\).

c) Find all the local maxima and local minima of \( f \).

**Answer:** The function has a local maximum at \( x = 0 \). The local maximum is 2. The local minimum is 0 at \( x = -\pi/2 \) and at \( x = \pi/2 \).

8. The derivative \( f' \) of a certain function \( f \) is given by

\[
f'(x) = -\frac{2x}{4 - x^2}.
\]

a) \( f''(x) = -\frac{2(x^2 + 4)}{(4 - x^2)^2} \). Since \( f'' < 0 \) for all \( x \) in the domain of \( f \), \( f \) is concave down everywhere in its domain.

**Answer:** The function is concave down on \((-\infty, -2), (-2, 2), (2, \infty)\). The function is nowhere concave up.

b) At which \( x \), if any, does \( f \) have an inflection point?

**Answer:** Since the graph of \( f \) does not change concavity, there are no inflection points.
9. A spherical snowball is melting. Its radius is decreasing at a rate of 0.2 cm per hour when the radius is 15 cm. How fast is its volume decreasing at that time?

Let
\[ x = \text{radius of the spherical snowball (in cm)}, \]
\[ V = \text{volume of snowball (in cubic centimeters)}, \]
\[ t = \text{time (in hours)}. \]

At anytime \( t \), the volume of the snowball is given by
\[ V = \frac{4}{3} \pi x^3 \ (*) \]
We know that when \( x = 15 \), the rate of change of the radius is \( \frac{dx}{dt} = -0.2 \). We want to find \( \frac{dV}{dt} \).

Differentiating (*) on both sides with respect to time \( t \), we get:
\[ \frac{dV}{dt} = 4\pi x^2 \frac{dx}{dt} \implies \frac{dV}{dt} = 4\pi (15)^2 (-0.2) = -180\pi. \]

**Answer**: The volume is *decreasing* at a rate of \( 180\pi \) cubic centimeters per hour.

10. An 8 feet by 15 feet cardboard will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What are the dimensions of the box with largest volume?

Let \( x = \text{height of the box (in feet)}, \) and \( f(x) = \text{volume of box (in cubic feet).} \) Note that in order to create a box, \( 0 < x < 4 \).

Then length = \( 15 - 2x \), and width = \( 8 - 2x \).
So, volume \( f(x) = x(15 - 2x)(8 - 2x) = 4x^3 - 46x^2 + 120x \), and
the derivative \( f'(x) \) is given by \( f'(x) = 12x^2 - 92x + 120 = 4(x - 6)(3x - 5). \)
The critical number is \( x = 5/3 \). (The \( x = 6 \) is NOT in the domain \((0, 4)\).)

The second derivative \( f''(x) = 24x - 92, \) and \( f''(5/3) < 0 \). So, by second derivative test \( f \) has a maximum at \( x = 5/3 \).

**Answer**: The volume is of the box is maximized when a 5/3 ft by 5/3 ft square is cut from the corners from the cardboard.