1. a) Simplify $\csc(\arctan(\sqrt{1+x^2}))$.

**Solution:**
Let $\theta = \arctan(\sqrt{1+x^2})$.
Then $\tan(\theta) = \frac{1+x^2}{1} = \frac{\text{opposite}}{\text{adjacent}}$.

Therefore, $\csc(\arctan(\sqrt{1+x^2})) = \csc(\theta) = \frac{\sqrt{2+x^2}}{\sqrt{1+x^2}}$.

b) Consider the function $f(x) = \ln \left( \frac{x}{x-1} \right)$. Find $f^{-1}(x)$.

**Solution:**

\[
x = \ln \frac{y}{y-1} \implies e^x = \frac{y}{y-1} \implies e^x(y-1) = y \implies y = \frac{e^x}{e^x-1}
\]

Therefore, $f^{-1}(x) = \frac{e^x}{e^x-1}$.

2. Evaluate the following limits. Show all necessary steps.

a) \[
\lim_{\theta \to 0} \frac{\tan^2(3\theta)}{\theta} = \lim_{\theta \to 0} \frac{\sin^2(3\theta)}{\theta \cos^2(3\theta)} = \lim_{\theta \to 0} \frac{1-\cos^2(3\theta)}{\theta \cos^2(3\theta)}
\]

\[
= \left( \lim_{\theta \to 0} \frac{1-\cos(3\theta)}{3\theta} \right) \left( 3 \lim_{\theta \to 0} \frac{1 + \cos(3\theta)}{\cos^2(3\theta)} \right)
\]

\[= 0 \left( 3 \cdot \frac{2}{1} \right) = 0.\]

b) \[
\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x - 2)(2x + 1)}{(x - 2)(x + 1)}
\]

\[= \lim_{x \to 2} \frac{2x + 1}{x + 1} = \frac{5}{3}.\]
3. Suppose that \( f(x) \) is a function such that \( f(3+h) - f(3) = e^{2h} - 1 \) for all \( h \neq 0 \).
   a) Find the slope of the secant line passing through the points \((3, f(3))\) and \((5, f(5))\).

Solution:

\[
\frac{\Delta f}{\Delta x} = \frac{f(5) - f(3)}{5 - 3} = \frac{f(3) + 2 - f(3)}{2} = \frac{e^{2} - 1}{2} = \frac{e^{4} - 1}{2}.
\]

b) Find the slope of the tangent line to the graph of \( f \) at \( x = 3 \).

Solution:

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{e^{2h} - 1}{h}
= \lim_{h \to 0} \frac{e^{2(0+h)} - e^{2(0)}}{h}
= g'(0), \quad \text{where } g(x) = e^{2x}.
\]

Therefore, \( \frac{df}{dx} \bigg|_{x=0} = 2e^{2x} \).

4. Suppose \( x = e^{2t} \cos t \) and \( y = e^{2t} \sin t \). Calculate \( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \), and simplify as much as possible.

Solution:

\[
\frac{dx}{dt} = 2e^{2t} \cos t - e^{2t} \sin t \quad \text{and} \quad \frac{dy}{dt} = 2e^{2t} \sin t + e^{2t} \cos t
\]

\[
\left( \frac{dx}{dt} \right)^2 = 4e^{4t} \cos^2 t - 4e^{4t} \cos t \sin t + e^{4t} \sin^2 t,
\]

and \( \left( \frac{dy}{dt} \right)^2 = 4e^{4t} \sin^2 t + 4e^{4t} \sin t \cos t + e^{4t} \cos^2 t \).

Therefore, \( \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = 4e^{4t} (\cos^2 t + \sin^2 t) + e^{4t} (\sin^2 t + \cos^2 t) \)

\[
\implies \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = 5e^{4t}.
\]
5. Let \( f(x) = \frac{1}{1 + x^2} \). Calculate \( f'(x) \) using the definition of the derivative.

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{1 + (x + h)^2} - \frac{1}{1 + x^2} = \lim_{h \to 0} \frac{(1 + x^2) - (1 + (x + h)^2)}{(1 + (x + h)^2)(1 + x^2)} \]

\[
= \lim_{h \to 0} \frac{(1 + x^2) - (1 + (x + h)^2)}{h(1 + x^2)(1 + (x + h)^2)} = \lim_{h \to 0} \frac{1 + x^2 - 1 - x^2 - h^2 - 2xh}{h(1 + x^2)(1 + (x + h)^2)} \]

\[
= \lim_{h \to 0} \frac{-h - 2x}{(1 + x^2)(1 + (x + h)^2)} = \frac{-2x}{(1 + x^2)^2}. \]

6. Consider the function

\[
f(x) = \begin{cases} \sin(x) & \text{if } x < \pi/3 \\ kx & \text{if } x \geq \pi/3 \end{cases}
\]

Find the value of \( k \) so that \( f \) is continuous everywhere.

Solution:

Note that \( y = \sin(x) \) and \( y = kx \) are continuous in their domain. So, we need to choose \( k \) so that \( f \) is also continuous at \( x = \pi/3 \).

We know, \( f \) is continuous at \( x = 3 \) if and only if

\[
\lim_{x \to \pi/3^-} f(x) = \lim_{x \to \pi/3^+} f(x) = f(\pi/3)
\]

We have \( \lim_{x \to \pi/3^-} f(x) = \lim_{x \to \pi/3^-} \sin x = \sin(\pi/3) = \sqrt{3}/2 \),

\[
\lim_{x \to \pi/3^+} f(x) = \lim_{x \to \pi/3^+} kx = k \frac{\pi}{3},
\]

and \( f(\pi/3) = \frac{k\pi}{3} \).

Therefore, \( \frac{k\pi}{3} = \sqrt{3}/2 \implies k = \frac{3\sqrt{3}}{2\pi} \).
7. One of those scary fish with a light on its head is moving along the $x$–axis according to the function

$$s(t) = \frac{1}{2}t^2 - 20\sqrt{t},$$

where $t$ is measured in minutes, and $s(t)$ is measured in leagues.

a) Compute the average velocity of the fish between $t = 0$ and $t = 4$.

**Solution:**

$$\text{Average velocity} = \frac{s(4) - s(0)}{4 - 0} = \frac{\left(\frac{1}{2}(4)^2 - 20\sqrt{4}\right) - (0)}{4} = -8 \text{ leagues per minute.}$$

b) Find the instantaneous velocity of the fish at time $t = 4$. Is the fish moving to the left or to the right at this time?

**Solution:**

$$\text{Instantaneous Velocity} = s'(4) = t - \frac{1}{2}(20)t^{-1/2}\bigg|_{t=4} = 4 - \frac{10}{2} = -1 \text{ leagues per minute.}$$

Therefore, the fish is moving to the left at a rate of 1 league per minute.

8. Consider the function $f(x) = 4x - 3$. We wish to prove that the limit $\lim_{x \to 2} f(x)$ exists.

a) If the limit exists, it must have what value?

$$L = \lim_{x \to 2} f(x) = 4(2) - 3 = 5.$$ 

b) In a formal proof that the limit exists, given a value of $\epsilon > 0$, we will use what quantity for $\delta$?

$$|f - L| = |(4x - 3) - 5| = |4x - 8| = 4|x - 2| < 4\delta = \epsilon. \quad \text{Thus, } \delta = \epsilon/4.$$ 

c) Complete the proof that $\lim_{x \to 2} f(x) = L$.

If $0 < |x - 2| < \delta$, then

$$|f - L| = |(4x - 3) - 5| = |4x - 8| = 4|x - 2| < 4\delta = 4\frac{\epsilon}{4} = \epsilon.$$
9. Use implicit differentiation to find an equation for the tangent line to the curve $3x^2 - 2xy - y^2 = 7$ at the point $(-2, 5)$.

Solution:

First, we will find the slope of the tangent line at $(-2, 5)$.

Differentiating both sides of the equation, we get

$$6x - 2y - 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{2y - 6x}{-2x - 2y}$$

So, the slope of the tangent line is given by

$$\left. \frac{dy}{dx} \right|_{(-2, 5)} = \frac{2y - 6x}{-2x - 2y} \bigg|_{(-2, 5)} = \frac{22}{-6} = -\frac{11}{3}.$$ 

Therefore, the equation of the tangent line to the graph of $f$ at $(-2, 5)$ is

$$y - 5 = -\frac{11}{3} (x + 2).$$