Study Guide for May 5 Final Exam, MATH251

New Topics covered for the exam: chapter 16, that is, sections 16.1, 16.2, 16.3, 16.4, 16.5, 16.6, 16.7 and 16.8. Remember that the final exam is cumulative, so you should also check the previous study guides and the 251 all sections site on Canvas for more information.

Disclaimer: the following guide contains a summary of the contents covered for this midterm. However, you should read the book since anything included there can be evaluated on the exam.

New things you need to know for the final exam

Section 16.1

- **Line integrals**: to integrate a continuous function \( f(x,y,z) \) over a curve \( C \): find a smooth parametrization of \( C \): \( \mathbf{r}(t) = g(t)i + h(t)j + k(t)k \), \( a \leq t \leq b \). The integral is evaluated as

\[
\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t)) \left| \mathbf{v}(t) \right| \,dt
\]

Section 16.2

- The **gradient** of a differentiable function \( f(x,y,z) \) is

\[
\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k
\]

- **Evaluating the line integral of** \( \mathbf{F} = Mi + Nj + Pk \) **along a curve** \( C \): find a parametrization of the curve \( \mathbf{r}(t) = g(t)i + h(t)j + k(t)k \), \( a \leq t \leq b \). Then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t),z(t)) \cdot \frac{d\mathbf{r}}{dt} \,dt
\]

When \( \mathbf{F} \) is interpreted as a force field, then the previous integral corresponds to the **work** done in moving an object from \( A = \mathbf{r}(a) \) to the point \( B = \mathbf{r}(b) \) along \( C \). If \( \mathbf{F} \) represents the velocity field of a fluid flow through a region in space, then
the previous integral corresponds to the flow along the curve from \( A = r(a) \) to the point \( B = r(b) \). If \( A = B \), the flow is called the circulation around the curve.

If \( C \) is a smooth simple closed curve in the domain of a continuous vector field \( \mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} \), and if \( \mathbf{n} \) is the outward pointing unit normal vector on \( C \), the flux of \( \mathbf{F} \) across \( C \) is

\[
\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint M \, dy - N \, dx
\]

Here \( x = g(t) \), \( y = h(t) \), \( a \leq t \leq b \), and the curve \( C \) is traveled counterclockwise exactly once.

**Section 16.3**

- Let \( \mathbf{F} \) be a vector field defined on an open region \( D \) in space, and suppose that for any two points \( A \) and \( B \) in \( D \) the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) along a path \( C \) from \( A \) to \( B \) in \( D \) is the same over all paths from \( A \) to \( B \). Then the integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independent in \( D \) and the field \( \mathbf{F} \) is conservative on \( D \).

- If \( \mathbf{F} \) is a vector field defined on \( D \) and \( \mathbf{F} = \nabla f \) for some scalar function \( f \) on \( D \), then \( f \) is called a potential function for \( \mathbf{F} \).

- A region \( D \) is called simply connected if every loop in \( D \) can be contracted to a point in \( D \) without ever leaving \( D \).

**Fundamental theorem of line integrals:** Let \( C \) be a smooth curve joining the point \( A \) to the point \( B \) in the plane or in space and parameterized by \( r(t) \). Let \( f \) be a differentiable function with a continuous gradient vector \( \mathbf{F} = \nabla f \) on a domain \( D \) containing \( C \). Then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)
\]

- Let \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) be a vector field whose components are continuous throughout an open connected region \( D \) in space. Then \( \mathbf{F} \) is conservative if and only if \( \mathbf{F} \) is a gradient field \( \nabla f \) for a differentiable function \( f \).

- \( \mathbf{F} \) is conservative on \( D \) if and only if \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \) around every loop (that is, closed curve \( C \)) in \( D \).

- Let \( \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, \( \mathbf{F} \) is conservative if and only if

\[
\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}
\]

**Section 16.4**

- The circulation density of a vector field \( \mathbf{F} = Mi + Nj \) at the point \((x,y)\) is the scalar expression \( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \). This expression is the k-component of the curl, denoted \((\text{curl}\mathbf{F}) \cdot \mathbf{k}\).
The divergence (flux density) of a vector field $F = M\mathbf{i} + N\mathbf{j}$ at the point $(x,y)$ is $\text{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

**Tangential Form of Green’s Theorem:** let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $F = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M, N$ having continuous first partial derivatives in an open region containing $R$. Then

$$\oint_C F \cdot T \, ds = \oint_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dy \, dx$$

**Normal Form of Green’s Theorem:** let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $F = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M, N$ having continuous first partial derivatives in an open region containing $R$. Then

$$\oint_C F \cdot n \, ds = \oint_C M \, dy - N \, dx = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dy \, dx$$

**Section 16.5**

A parametrization of a surface is given by three parametric equations

$$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$$

which can be described in terms of the vector $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$

One defines

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = f_u\mathbf{i} + g_u\mathbf{j} + h_u\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = f_v\mathbf{i} + g_v\mathbf{j} + h_v\mathbf{k}$$

The surface area of a smooth surface given by a parametrization $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is $[a \leq u \leq b, \ c \leq v \leq d]$

$$A = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

The surface area of a graph $z = f(x,y)$ over a region $R$ is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dy \, dx$$

**Some Standard Parametrizations of Surfaces:**

These are some common parametrizations of surfaces. It is important to observe that the same surface can be parametrized in different ways, so this is not an exhaustive list.
Cylinder of radius \( R \) with \( z \)-axis as central axis: \( x^2 + y^2 = R^2 \)

\[
\begin{align*}
\mathbf{r}(\theta, z) &= (R \cos \theta, R \sin \theta, z) \quad \text{parametrization} \\
\mathbf{r}_\theta \times \mathbf{r}_z &= \langle R \cos \theta, R \sin \theta, 0 \rangle \quad \text{outward normal vector} \\
dS &= \| \mathbf{r}_\theta \times \mathbf{r}_z \| \; dz \; d\theta = R \; dz \; d\theta \quad \text{surface element}
\end{align*}
\]

Sphere of radius \( R \), centered at the origin: \( x^2 + y^2 + z^2 = R^2 \)

\[
\begin{align*}
\mathbf{r}(\theta, \phi) &= R(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \quad \text{parametrization} \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= R^2 \sin \phi \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle \quad \text{outward normal vector} \\
dS &= \| \mathbf{r}_\phi \times \mathbf{r}_\theta \| \; d\phi \; d\theta = R^2 \sin \phi \; d\phi \; d\theta \quad \text{surface element}
\end{align*}
\]

Graph of \( z = f(x, y) \):

\[
\begin{align*}
\mathbf{r}(x, y) &= (x, y, f(x, y)) \quad \text{parametrization} \\
\mathbf{r}_x \times \mathbf{r}_y &= \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle \quad \text{outward normal vector} \\
dS &= \| \mathbf{r}_x \times \mathbf{r}_y \| \; dy \; dx = \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \; dy \; dx \quad \text{surface element}
\end{align*}
\]

Section 16.6

If \( G(x, y, z) \) is defined on the surface \( S \) defined as \( \mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k} \), then

\[
\iint_S G(x, y, z) d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dudv
\]

where \( u, v \) take values over the region \( R \).

If the surface \( S \) is given as the graph of \( z = f(x, y) \), then

\[
\iint_S G(x, y, z) d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dy dx
\]

Let \( \mathbf{F} \) be a vector field in three-dimensional space with continuous components defined over a smooth surface \( S \) having a chosen field of normal unit vectors \( \mathbf{n} \) orienting \( S \). Then the surface integral of \( \mathbf{F} \) over \( S \) is

\[
\oiint_S \mathbf{F} \cdot \mathbf{n} d\sigma
\]

where \( \mathbf{n} \) is a unit normal vector which orients \( S \). When the surface is parameterized as \( \mathbf{r}(u, v) \), then

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}
\]

and

\[
\oiint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| \; du \; dv = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \; du \; dv
\]
Section 16.7

\( \nabla \) \( \times \) \( \mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k} \) is

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M & N & P
\end{vmatrix}
= \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \hat{i} + \left( \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \hat{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}
\]

\( \nabla \times \mathbf{F} \)

Stokes’ Theorem: let \( S \) be a piecewise smooth oriented surface having a piecewise smooth boundary curve \( C \). Let \( \mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k} \) be a vector field whose components have continuous first partial derivatives on an open region containing \( S \). Then the circulation of \( \mathbf{F} \) around \( C \) in the direction counterclockwise with respect to the surface’s unit normal vector \( \mathbf{n} \) equals the integral of the curl vector field \( \nabla \times \mathbf{F} \) over \( S \):

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, d\sigma
\]

or in a different notation

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, d\sigma
\]

Important identity: for any scalar-valued function \( f(x, y, z) \), \( \nabla \times \nabla f = 0 \).

If \( \nabla \times \mathbf{F} = 0 \) at every point of a simply connected open region \( D \) in space, then on any piecewise-smooth closed path \( C \) in \( D \), \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \).

Section 16.8

\( \nabla \cdot \mathbf{F} \)

\( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \)

Divergence Theorem: let \( \mathbf{F} \) be a vector field whose components have continuous first partial derivatives, and let \( S \) be a piecewise smooth oriented closed surface. The flux of \( \mathbf{F} \) across \( S \) in the direction of the surface’s outward unit normal field \( \mathbf{n} \) equals the triple integral of the divergence \( \nabla \cdot \mathbf{F} \) over the region \( D \) enclosed by the surface

\[
\iiint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV
\]

or in a different notation

\[
\iiint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \text{div}(\mathbf{F}) \, dV
\]

Important identity: for any differentiable vector field \( \mathbf{F}(x, y, z) \): \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \).