This examination booklet contains 6 questions on 10 pages of paper including the front cover.

Do all of your work in this booklet, show all your computations and justify/explain your answers.

Do not remove any pages.

Your justification must be based on techniques already discussed in this course. If asked to evaluate an integral, remember to show all the steps that gives you its value.

Except for your personal note sheet, no other resources like class notes, books, calculator, etc are allowed. Remember that your note sheet must be handwritten, on both sides of a single sheet of paper.

Unless otherwise stated, give exact answers. For example, write π and √2 instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of e^0, and you must write π/3 instead of sec^{-1}(2).

If you run out of space when answering a problem, you may use any of the last three pages of the exam, but you must: indicate in the space below the question that you are continuing your answer on the extra sheet, and indicate on the extra sheet which problem you are working on.

If asked to use a specific theorem to calculate some quantity, you will receive no points if that theorem is not used.

Do not discuss the exam with anyone until grades are posted on Canvas.

WRITE OUT AND SIGN PLEDGE

On my honor, I have neither received nor given any unauthorized assistance on this examination.
Problem 1. [* points] Suppose $\mathbf{u}, \mathbf{v}$ are two vectors such that $\mathbf{u} \cdot \mathbf{v} = 2$, $\mathbf{u} \cdot \mathbf{u} = 4$, $\mathbf{v} \cdot \mathbf{v} = 4$. Then find:

a) $|\mathbf{u}|$

b) $(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$

c) The angle in radians between $\mathbf{u}$ and $\mathbf{v}$

d) The area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

**sol**

a) $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{4} = 2$

b) 

\[
(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 3\mathbf{u} \cdot \mathbf{u} + 3\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = 3(4) + 2\mathbf{u} \cdot \mathbf{v} - 4 = 12 + 4 - 4 = 12
\]

c) $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{1}{2}$ thus $\theta = \frac{\pi}{3}$

d) The area is $A = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta = 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$
Problem 2. [* points] Consider the points \( P = (a, -4, -4), Q = (-5, -4, -5) \) and \( R = (-1, -1, -3) \).

a) Find the value of \( a \) so that the plane containing \( P, Q, R \) is parallel to the plane \( 3x + 16y - 30z = 68 \).

b) Use the value of \( a \) bound in a) to write an equation of the plane containing the points \( P, Q, R \).

**sol:**

The normal vector can be taken as

\[
\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 - a & 0 & -1 \\ -1 - a & 3 & 1 \end{pmatrix} = \langle 3, 6 + 2a, -15 - 3a \rangle
\]

We want this vector to be parallel to \( \langle 3, 16, -30 \rangle \). By inspection we see that \( a = 5 \) is needed.

The equation of the plane becomes

\[
3x + 16y - 30z = 71
\]
Problem 3. [* points] Consider the curve given by the equation $r(t) = (t^3 + 1)i + (t^2 - 8t)j$. Use this information to answer the following items (which are independent of each other).

a) Find the value(s) of $t$ where the velocity vector of $r(t)$ is orthogonal (perpendicular) to the vector $\overrightarrow{PQ}$, where $P = (2, -1)$ and $Q = (4, 5)$.

b) Compute the unit tangent vector $T(t)$.

sol

a) $v(t) = \langle 3t^2, 2t - 8 \rangle$, $\overrightarrow{PQ} = \langle 2, 6 \rangle$, $v \cdot \overrightarrow{PQ} = 6(t^2 + 2t - 8) = 6(t + 4)(t - 2)$ so we need $t = -4$ and $t = 2$.

b) $T(t) = \frac{v(t)}{|v(t)|} = \frac{\langle 3t^2, 2t - 8 \rangle}{\sqrt{9t^4 + (2t - 8)^2}}$
Problem 4. [* points] Suppose that the velocity of a particle is given by \( \mathbf{v}(t) = \langle 2e^t, 3\pi \cos(3\pi t), \frac{2}{\sqrt{1+t}} \rangle \), \( -1 < t < \infty \).

a) Find the position \( \mathbf{r}(t) \) of the particle given that \( \mathbf{r}(0) = \langle 1, 2, 3 \rangle \).
b) Find the acceleration \( \mathbf{a}(t) \) of the particle.

sol

a) \[
\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(u) du
= \left\langle 1 + \int_0^t 2e^u du, 2 + \int_0^t 3\pi \cos(3\pi u) du, 3 + \int_0^t \frac{2}{\sqrt{1+u}} du \right\rangle
= \left\langle 2e^t - 1, \sin(3\pi t) + 2, 4\sqrt{1+t} - 1 \right\rangle
\]
b) \( \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \left\langle 2e^t, -9\pi^2 \sin(3\pi t), -\frac{1}{(1+t)^{3/2}} \right\rangle \)
Problem 5. [∗ points] The following question are independent of one another.

a) Can two different level curves of a function $f(x, y)$ intersect or not.

b) Consider the function $f(x, y, z) = \frac{4-z^2}{x^2+y^2}$. Write an equation for the level surface $f(x, y, z) = 1$ in such a way that no fractions appear. Identify the surface that you get: for example, is it a sphere, cone, paraboloid, hyperboloid?

sol:

a) A level curve is where $f(x, y) = c$ for some constant $c$. If $(x_0, y_0)$ belongs to two different level curves $f(x, y) = c_1, f(x, y) = c_2$, then $f(x_0, y_0) = c_1 = c_2$ and therefore $c_1 = c_2$, which can’t happen since the level curves were related to different values.

b) The level surface equation becomes $4 - z^2 = x^2 + y^2$, or $4 = x^2 + y^2 + z^2$, which is the equation of a sphere.
Problem 6. [* points] Consider the function \( f(x, y) = \frac{x + y}{\sqrt{x^2 + y^2}} \)

a) Find the domain of the function.
b) Find \( \lim_{(x,y)\to(0,0)} f(x, y) \) or demonstrate that the limit does not exist.

sol

a) we need \( x^2 + y^2 > 0 \), which is true for all points \( (x, y) \) except the origin \( (0, 0) \). Therefore, the domain is the \( xy \) plane with the origin removed.
b) We use the path test: first path is \( y = x \) along the first quadrant

\[
\lim_{(x,y)\to(0,0)} \frac{x + y}{\sqrt{x^2 + y^2}} = \lim_{x\to0^+} \frac{2x}{\sqrt{2x^2}} = \frac{2}{\sqrt{2}} \lim_{x\to0^+} \frac{x}{x} = \frac{2}{\sqrt{2}}
\]

Path \( y = 0 \) along positive \( x \) axis

\[
\lim_{(x,y)\to(0,0)} \frac{x + y}{\sqrt{x^2 + y^2}} = \lim_{x\to0^+} \frac{2x}{\sqrt{2x^2}} = \frac{2}{\sqrt{2}} \lim_{x\to0^+} \frac{x}{x} = \frac{2}{\sqrt{2}}
\]

Since the answer depends on the path , the overall limit cannot exist by the path test.
YOU CAN USE THIS PAGE FOR ANY QUESTION OF THE EXAM OR SCRATCH-WORK:
YOU CAN USE THIS PAGE FOR ANY QUESTION OF THE EXAM OR SCRATCHWORK:
YOU CAN USE THIS PAGE FOR ANY QUESTION OF THE EXAM OR SCRATCH-WORK: