EXAM 1-B FOR MATH-251

OCTOBER 10, 2022 (80 MINUTES)

Name [please print]:

NetId:

Instructor’s Name:

Section:

This examination booklet contains 6 questions on 10 pages of paper including the front cover.

Do all of your work in this booklet, show all your computations and justify/explain your answers.

Do not remove any pages.

Your justification must be based on techniques already discussed in this course. If asked to evaluate an integral, remember to show all the steps that gives you its value.

Except for your personal note sheet, no other resources like class notes, books, calculator, etc are allowed. Remember that your note sheet must be handwritten, on both sides of a single sheet of paper.

Unless otherwise stated, give exact answers. For example, write π and \( \sqrt{2} \) instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of \( e^0 \), and you must write \( \pi/3 \) instead of \( \sec^{-1}(2) \).

If you run out of space when answering a problem, you may use any of the last three pages of the exam, but you must: indicate in the space below the question that you are continuing your answer on the extra sheet, and indicate on the extra sheet which problem you are working on.

If asked to use a specific theorem to calculate some quantity, you will receive no points if that theorem is not used.

Do not discuss the exam with anyone until grades are posted on Canvas.

You may solve the exam problems in any order you wish.

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WRITE OUT AND SIGN PLEDGE

On my honor, I have neither received nor given any unauthorized assistance on this examination.
Problem 1. [18 points] The following two questions are independent of one another.

a) Suppose that $\mathbf{u}, \mathbf{v}$ are two vectors such that $\mathbf{v} \times \mathbf{u} = i - 3j + 5k$. Using the properties of the cross product, find

$$ (3\mathbf{u} + 2\mathbf{v}) \times (\mathbf{u} - 7\mathbf{v}) $$

**sol**

We have

$$ (3\mathbf{u} + 2\mathbf{v}) \times (\mathbf{u} - 7\mathbf{v}) \\
= 3\mathbf{u} \times \mathbf{u} - 21\mathbf{u} \times \mathbf{v} + 2\mathbf{v} \times \mathbf{u} - 14\mathbf{v} \times \mathbf{v} \\
= 21\mathbf{v} \times \mathbf{u} + 2\mathbf{v} \times \mathbf{u} \\
= 23\mathbf{v} \times \mathbf{u} \\
= 23i - 69j + 115k $$

b) Suppose $P = (4, 1, 2), Q = (-2, 3, 5)$. If $\mathbf{w} = \langle a, 7, b \rangle$ is parallel to $\overrightarrow{PQ}$, find $a$ and $b$.

**sol**

The vector $\overrightarrow{PQ}$ equals

$$ \overrightarrow{PQ} = \langle -2, 3, 5 \rangle - \langle 4, 1, 2 \rangle = \langle -6, 2, 3 \rangle $$

If $\mathbf{w}$ is parallel to $\overrightarrow{PQ}$, then $\overrightarrow{PQ} = \lambda \mathbf{w}$ for some scalar $\lambda$. This gives the system

$$ \begin{align*}
-6 &= \lambda a \\
2 &= 7\lambda \\
3 &= \lambda b
\end{align*} $$

The second equation says that $\lambda = \frac{2}{7}$, and substituting in the other two equations we get $a = -\frac{6}{\frac{2}{7}} = -21$ and $b = \frac{3}{\frac{2}{7}} = \frac{21}{2}$.

Alternatively, just check for what values of $a, b$ you get $\overrightarrow{PQ} \times \mathbf{w} = \mathbf{0}$, and you find the same answer.
Problem 2. [18 points] Find an equation of the plane containing the line \( \mathbf{r}(t) = \langle 2 + 7t, 5 + 3t, 4 - 6t \rangle \) and the point \( P = (3, -1, 6) \).

\textbf{sol}

A point on the line is \( \mathbf{r}(0) = (2, 5, 4) \), and a vector giving the direction of the line is \( \frac{d\mathbf{r}}{dt} = \langle 7, 3, -6 \rangle \).

The vector \( \overrightarrow{\mathbf{r}(0)P} = \langle 3, -1, 6 \rangle - \langle 2, 5, 4 \rangle = \langle 1, -6, 2 \rangle \) is another vector lying on the plane. Thus, we can take the normal vector as

\[
\mathbf{n} = \frac{d\mathbf{r}}{dt} \times \overrightarrow{\mathbf{r}(0)P} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
7 & 3 & -6 \\
1 & -6 & 2
\end{vmatrix}
= \langle -30, -20, -45 \rangle
\]

Finally, the plane equation is

\[
-30(x - 2) - 20(y - 5) - 45(z - 4) = 0
\]

Or equivalently

\[
30x + 20y + 45z = 340
\]
Problem 3. [20 points] The following questions are independent of one another:

a) Suppose that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ represent the parametrizations of two curves. Find

$$ \frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) \bigg|_{t=0} $$

if $\mathbf{r}_1(t) = \langle t^2 + 5, 3\sqrt{t+9}, 4+3t \rangle$, $\mathbf{r}_2(0) = \langle 4, -3, 2 \rangle$ and $\frac{d\mathbf{r}_2}{dt} \big|_{t=0} = \langle 7, 1, -5 \rangle$.

sol

Use the product rule

$$ \frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) \bigg|_{t=0} $$

$$ = \frac{d}{dt}(\mathbf{r}_1(t)) \bigg|_{t=0} \cdot \mathbf{r}_2(0) + \mathbf{r}_1(0) \cdot \frac{d}{dt}(\mathbf{r}_2(t)) \bigg|_{t=0} $$

$$ = \left\langle 2t, \frac{3}{2\sqrt{t+9}}, 3 \right\rangle \bigg|_{t=0} \cdot \langle 4, -3, 2 \rangle + \langle 5, 9, 4 \rangle \cdot \langle 7, 1, -5 \rangle $$

$$ = \left\langle 0, \frac{1}{2}, 3 \right\rangle \cdot \langle 4, -3, 2 \rangle + \langle 5, 9, 4 \rangle \cdot \langle 7, 1, -5 \rangle $$

$$ = 0 - \frac{3}{2} + 6 + 35 + 9 - 20 $$

$$ = \frac{57}{2} $$

b) A lanternfly with velocity vector $\mathbf{r}'(t)$ starts at the origin at time $t = 0$ and flies around for $k$ seconds.

b.1) Where is the lanternfly at time $k$ if $\int_0^k \mathbf{r}'(t) dt = 0$?

sol

We have

$$ \mathbf{r}(k) = \mathbf{r}(0) + \int_0^k \mathbf{r}'(t) dt = \mathbf{r}(0) + 0 = \mathbf{r}(0) $$

Thus the lanternfly is located where it started.
b.2) What does the quantity $\int_0^k |r'(t)| \, dt$ represent?

**sol**

$\int_0^k |r'(t)| \, dt$ represents the distance the lanternfly traveled in completing its path.
Problem 4. [16 points] Evaluate the limit  
\[
\lim_{(x,y) \to (0,0)} \frac{x^4 - 6x^2y^2 + y^4}{(3x^2 + 3y^2)^2}
\]
or show that it does not exist. Justify your answer either way.

\textbf{sol}

Consider moving along the path \( y = x \):

\[
\lim_{(x,y) \to (0,0)} \frac{x^4 - 6x^2y^2 + y^4}{(3x^2 + 3y^2)^2} = \lim_{x \to 0} \frac{x^4 - 6x^4 + x^4}{(6x^2)^2} = \frac{-4}{36} = -\frac{1}{9}
\]

Now consider moving along the path \( y = 2x \):

\[
\lim_{(x,y) \to (0,0)} \frac{x^4 - 6x^2y^2 + y^4}{(3x^2 + 3y^2)^2} = \lim_{x \to 0} \frac{x^4 - 24x^4 + 16x^4}{(15x^2)^2} = -\frac{7}{15^2}
\]

Since the answer for the limit depends on the path chosen, by the two path test it cannot exist.
Problem 5. [18 points] a) Find the domain of the function

\[ f(x, y) = \frac{\sqrt{x + 2y}}{\sqrt{4x - y}} \]

sol
The domain consists of all the points \((x, y)\) such that \(x + 2y \geq 0\) and \(4x - y > 0\).

b) Sketch the domain of \(f(x, y)\) on the xy plane.

in this picture the origin should be excluded

\[ \text{NOT DOMAIN} \quad \text{DOMAIN} \]

\[ \text{NOT DOMAIN} \quad \text{NOT DOMAIN} \]

\[ y = x \]

This is the equation of a line

but you must only draw the part of the line contained in the domain found in part b)

c) Sketch level curve \(f(x, y) = 1\).

sol
We must sketch \(\sqrt{x + 2y} = \sqrt{4x - y}\). Square both sides to obtain \(x + 2y = 4x - y\) or \(3y = 3x\), that is, \(y = x\). This is the equation of a line
Problem 6. [20 points] Let $C$ be the curve corresponding to the intersection of the surfaces with equations $z^2 = (x - 2)^2 + y^2$ and $x^2 + y^2 = 16$.

a) Sketch the surfaces $z^2 = (x - 2)^2 + y^2$ and $x^2 + y^2 = 16$. You can graph them together or separately but, you must label which equation corresponds to which surface.

sol:
the surfaces are a cone with vertex at $(2, 0, 0)$ and a cylinder of radius 4 with the $z$ axis as the axis of symmetry.

b) Find a parametrization $\mathbf{r}(t)$ for the part of $C$ which lies inside the first octant. Specify the values the parameter $t$ takes.

The equation $x^2 + y^2 = 16$ gives $x = 4 \cos t$, $y = 4 \sin t$. Then

$$z = \sqrt{(x - 2)^2 + y^2} = \sqrt{(4 \cos t - 2)^2 + 16 \sin^2 t}$$

And the parametrization becomes

$$\mathbf{!}(t) = \left( 4 \cos t, 4 \sin t, \sqrt{(4 \cos t - 2)^2 + 16 \sin^2 t} \right)$$

with $0 \leq t \leq \frac{\pi}{2}$ since we want the part of the curve inside the first octant.

c) Find the equation of the tangent line to the curve at the point $P = (2, 2\sqrt{3}, 2\sqrt{3})$.

sol
Observe that $(2, 2\sqrt{3}, 2\sqrt{3}) = \mathbf{r}(\pi/3)$. Moreover

$$\mathbf{v}(t) = \left\langle -4 \sin t, 4 \cos t, \frac{2(4 \cos t - 2)(-4 \sin t) + 32 \sin t \cos t}{2\sqrt{(4 \cos t - 2)^2 + 16 \sin^2 t}} \right\rangle$$

Thus

$$\mathbf{v}(\pi/3) = \left\langle -2\sqrt{3}, 2, 2 \right\rangle$$

And the parametric equations for the line become

$$\begin{align*}
x &= 2 - 2\sqrt{3}t \\
y &= 2\sqrt{3} + 2t \\
z &= 2\sqrt{3} + 2t
\end{align*}$$
YOU CAN USE THIS PAGE FOR ANY QUESTION OF THE EXAM OR SCRATCH-WORK:
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