RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2020

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that for any pair of commuting $n \times n$-matrices with complex entries there exists a common eigenvector.

2. Prove that there exists no simple group of order 56.

3. Prove that a ring which contains a principal ideal ring R, and which is contained in the field of fractions of R, is a principal ideal ring.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let A and B be two projection linear maps in a vector space over a field K. Prove that if $A + B$ is a projection linear map and $charK \neq 2$ then $AB = BA = 0$.

5. Prove that in the group $\mathbf{Q}/\mathbf{Z}$ for any natural number n there exists exactly one subgroup of order n.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x+x^2} dx$ using Cauchy's calculus of residues.

2. Prove that if $1 < a < \infty$ is a real number, then $f_a(z) = z + a - e^z$ has only one zero in the half-plane $\{z : Re(z) < 0\}$ and that the zero is real.

3. Construct a conformal map from the half-strip $D := \{z \in \mathbb{C} | Re(z) < 0, 0 < Im(z) < 1\}$ to the upper-half plane such that it has a continuous extension to the closure of D considered as a map to the extended complex plane, and maps 0 to 0. You may construct the map as the composition of several elementary conformal maps.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $f(z)$ be an injective holomorphic function on the punctured disk $D_0 = \{z : 0 < |z| < 1\}$ such that the area of its mapping image $f(D_0)$ is finite. Prove that the length of the image by f of the interval $(0, 1/2]$ on the x-axis is finite.

5. Let f be holomorphic on a neighborhood of the closed unit disc centered at the origin. Assume that $|f(z)| = 1$ if $|z| = 1$, and is not a constant on the disc. Prove that there exist a positive integer k, points $\alpha_1, \ldots, \alpha_k$ in the open unit disc, positive integers $m_1, \ldots, m_k$, and a complex number β with $|\beta| = 1$ such that

$$f(z) = \beta \prod_{l=1}^{k} \left(\frac{z - \alpha_l}{1 - \bar{\alpha}_l z} \right)^{m_k} \qquad \text{for all } z \text{ in the unit disc.}$$

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Written Qualifying Examination

August 2020

Session 3. Real Variables and Elementary Point-Set Topology Written Qualifying Exam

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let E_k be a sequence of Lebesgue measurable subsets of $\mathbb{R}$. Let

$$E = \{x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k\}.$$

1. Show that E is Lebesgue measurable.
2. Show that if $\sum_{i=1}^{\infty} |E_k| < \infty$, then $|E| = 0$.
3. Assume instead only that $\lim_{k\to\infty} |E_k| = 0$. Must $|E| = 0$?

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of integrable functions with $\int_{\mathbb{R}} |f_n| \leqslant 1$. Assume that there exists a measurable function f such that $|\{x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon\}| \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

1. Show that there is a subsequence f_{n_k} which converges to f almost everywhere.
2. Show that f is integrable.

3. Let M be a metric space, N be a complete metric space, and $S \subset M$ is a dense subset. Let $f : S \to N$ be a uniformly continuous function. Show that there exists a unique continuous function $g : M \to N$ such that $g|_S = f$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Construct a nondecreasing function $f : (0,1) \to \mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q} \cap (0,1)$ (the rational numbers in $(0,1)$), or prove that such a function does not exist.

5. Let $f(x,y) = \frac{xy}{(|x|+|y|)^\alpha}$ for $(x,y) \neq 0$, and $f(0,0) = 0$, where $\alpha \in \mathbb{R}$. For what values of α is f integrable on $(-1,1) \times (-1,1)$? Justify your answer.

End of Session 1