1. (6 points each) Find the following limits, giving reasons for your answers. You may use any method from this course.

a. \( \lim_{x \to 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}} = \frac{2 \sqrt{3}}{1} = 2 \sqrt{3} \)

\( \text{OR \ multiply \ by \ the \ conjugate, \ etc.} \)

b. \( \lim_{x \to 0} \frac{\cos 2x - 1}{x^3} = -2 \)

\( \text{OR} \quad \lim_{x \to 0} \frac{-2 \sin 2x}{2x} = -2 \lim_{x \to 0} \frac{\sin 2x}{2x} = -2 \cdot 1 = -2 \)

c. \( \lim_{x \to \infty} x \sin \left(\frac{2}{x}\right) = \frac{+2}{x} \)

\( \text{OR} \quad \lim_{x \to \infty} \frac{\sin \left(\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\cos \left(\frac{2}{x}\right)}{-\frac{1}{x^2}} = \cos(0) \cdot (1) = +2 \)
2. (9 points each) Find the derivatives of the following functions. You do not need to simplify your answers.

a. If \( y = \cos^3 x \sin(x^5) \) then \( \frac{dy}{dx} = \frac{\frac{d}{dx} (\cos^3 x) \sin(x^5) + \cos^3 x \frac{d}{dx} \sin(x^5)}{(1 + \ln x)^2} = 3 \cos^2 x (-\sin x) (\sin x^5) + \cos^3 x \cdot \cos x^5 \cdot 5x^4 \)

b. If \( y = \frac{x \sin x}{1 + \ln x} \), then \( \frac{dy}{dx} = \frac{(1 + \ln x) (\cos x + x \cos x) - x \sin x (\frac{1}{x})}{(1 + \ln x)^2} = \frac{x \cos x + x \cos x \ln x + \sin x \ln x}{(1 + \ln x)^2} \)
3. (9 points each) Find the following indefinite integrals.

a. \[ \int \frac{t^2 + 5t + 1}{\sqrt{t}} \, dt = \]

\[
= \int \left( \frac{t^2}{\sqrt{t}} + \frac{5t}{\sqrt{t}} + \frac{1}{\sqrt{t}} \right) \, dt \\
= \int \left( t^{3/2} + 5t^{1/2} + t^{-1/2} \right) \, dt \\
= \frac{t^{5/2}}{\frac{5}{2}} + \frac{5t^{3/2}}{\frac{3}{2}} + \frac{t^{1/2}}{\frac{1}{2}} + C
\]

b. \[ \int \sin(\cos x) \sin x \, dx = \]

\[
= \int \sin u \, (-du) = -\cos u + C \\
= \cos(\cos x) + C
\]
4. (9 points each).

a. \[ \int_{3}^{4} (1+e^{x})^5 e^{x} \, dx = \quad \text{Do not try to simplify your answer!} \]

\[
\begin{align*}
\text{Let } u &= 1 + e^{x} \\
\text{du} &= e^{x} \, dx \\
\int_{1+e^{3}}^{1+e^{4}} u^5 \, du &= \frac{u^6}{6} \bigg|_{1+e^{3}}^{1+e^{4}} \\
&= \frac{1}{6} \left[ (1+e^{4})^6 - (1+e^{3})^6 \right]
\end{align*}
\]

b. \[ \int_{2}^{3} x \sqrt{x-1} \, dx = \]

\[
\begin{align*}
\text{Let } u &= x-1 \\
\text{du} &= dx \\
\int_{1}^{2} (u+1) u^{1/2} \, du &= \int_{1}^{2} (u^{3/2} + u^{1/2}) \, du \\
&= \left[ \frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2} \right]_{1}^{2} \\
&= \frac{2^{5/2}}{5/2} - \frac{1}{5/2} + \frac{2^{3/2}}{3/2} - \frac{1}{3/2}
\end{align*}
\]
5. (17 points) Let \( f(x) = g(x^3 - 5) \). It is impossible to find \( g(x) \), but a few values of \( g(x) \) and \( g'(x) \) are known: \( g(1) = 2 \), \( g(2) = 5 \), \( g(3) = 7 \), \( g(4) = 2 \), \( g(5) = 11 \), \( g(6) = 13 \), \( g(7) = 21 \), \( g'(1) = 3 \), \( g'(2) = 2 \), \( g'(3) = 8 \), \( g'(4) = 10 \), \( g'(5) = 12 \), \( g'(6) = 21 \) and \( g'(7) = 23 \).

   a. Find \( f(2) \).

   \[
   f(2) = g(2^3 - 5) = g(3) = 7
   \]

   b. Find \( f'(2) \).

   \[
   f'(x) = g'(x^3 - 5) \cdot 3x^2
   \]

   \[
   f'(2) = g'(3) \cdot 12 = 8 \cdot 12 = 96
   \]

6. (18 points) Find the equation of the tangent line to the curve described by

\[
x^3y + xy^3 + x^2y - 2x^2 = -2
\]

at the point \((1, 0)\). Any correct equation specifying this line is acceptable.

\[
\text{Tangent line: } (y - 0) = 2 \left( x - 1 \right)
\]

\[
3x^2y + x^3y' + y^3 + 3xy^2y' + 2xy + x^2y' - 4x = 0
\]

Plug in \( x = 1, y = 0 \) \( \Rightarrow y' \)

\[
y' - 4 = 0
\]

\[
y' = 2
\]

\[
\frac{dy}{dx} = -\frac{3x^2y + y^3 + 2xy - 4x}{x^3 + 3xy^2 + x^2}
\]

If you want to work out \( \frac{dy}{dx} \) for all \( x \) and \( y \).
7. (18 points) The tangent line to \( y = f(x) \) at \( x = 2 \) is given by \( y = 7x + 3 \).
   a. (5 points) What is \( f'(2) \)? \( f' \)
   b. (5 points) What is \( f(2) \)? \( f(2) \)
   c. (6 points) Use linear approximation to approximate \( f(2.1) \).
   d. (2 points) If \( f''(x) < 0 \) for all \( x \), is this approximation too large or too small? \( \text{Too LARGE} \)

8. (17 pts) Find the absolute maximum and minimum of the function \( f(x) = x^3 + 3x^2 - 9x \) on the interval \([-2, 2]\).

<table>
<thead>
<tr>
<th>Absolute max:</th>
<th>((2, 22))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute min:</td>
<td>((1, -5))</td>
</tr>
</tbody>
</table>

\[
f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x+3)(x-1)
\]

\( C \) \( (1) \#S: \ x = \frac{-3}{2}, 1 \)

\[
\begin{align*}
f(-2) & = 22 \\
f(1) & = -5 \\
f(2) & = -2
\end{align*}
\]
9. (6 points each)
a. Find \( \frac{dy}{dx} \) if \( y = x^{8e} \).

\[
\ln y = 8x \ln x
\]

\[
\frac{1}{y} \frac{dy}{dx} = 8 \ln x + 8 \frac{1}{x}
\]

\[
y' = x^{8e} \left[ 8 \ln x + 8 \right]
\]

b. Find \( \frac{dy}{dx} \) if \( y = \int_0^x \sin t^2 \, dt \).

\[
\frac{dy}{dx} = \sin x^2
\]

c. Find \( \frac{dy}{dx} \) if \( y = \int_0^{x^2} \sin t^2 \, dt \).

\[
\frac{dy}{dx} = \sin (x^4)^2 \cdot 2x
\]

\[
= (\sin x^4) \cdot 2x
\]
10. (18 points) A mad scientist sells radioactive bats to her friends. Experience tells her that she will sell 20 bats per month if she charges 30 dollars per bat and that each $2 decrease in price will result in four more sales per month. How much should she charge per bat to maximize her revenue?

| Price per bat: | $20 |

\[
R(x) = (30-2x)(20+4x)
\]

\[
R'(x) = -2(20+4x) + (80-2x) \cdot 4
\]

\[
= -80 - 8x + 120 - 8x
\]

\[
= 80 - 16x = 0 \Rightarrow x = 5
\]

She sells 40 bats at $20 each.
11. (18 points) A rectangular poster is to contain 80 square inches of printed matter with 5 inch margins at the top and bottom and 4 inch margins at the sides. If posterboard costs 10 cents per square inch, what are the dimensions of the least expensive poster satisfying the requirements?

\[
(x-8)(y-10) = 80
\]

\[
y = \frac{80}{x-8} + 10 = \frac{10x}{x-8}
\]

\[
A = xy = 10x^2
\]

\[
A' = \frac{(x-8)(20x) - 10x^2}{(x-8)^2}
\]

\[
= \frac{20x^2 - 160x - 10x^2}{(x-8)^2} = 0
\]

\[
= \frac{10x^2 - 160x}{(x-8)^2}
\]

\[
x = 16
\]

\[
y = 10 \cdot \frac{16}{8} = 20
\]
12. (18 points) Compute the value of the Riemann sum for the function $f(x) = x^2$ on the interval $[1, 3]$ using $n = 4$ and taking $x_k^*$ to be the midpoint of the $k^{th}$ interval in the partition. You can leave your answer as a sum of fractions.

Value: 

\[
\frac{1}{2} \left[ \left( \frac{1}{2} \right)^2 + \left( \frac{3}{4} \right)^2 + \left( \frac{5}{4} \right)^2 + \left( \frac{3}{2} \right)^2 \right]
\]

13. (18 points) The length of a rectangle is decreasing at 4 in/min and its width is increasing at 5 in/min. How fast is the length of the diagonal changing when the length is 8 in and the width is 6 in?

\[
\text{length} = l, \quad \text{width} = w
\]

\[
2^2 = l^2 + w^2
\]

\[
2 \frac{dl}{dt} = 2l \frac{dl}{dt} + 2w \frac{dw}{dt}
\]

\[
\frac{dl}{dt} = \frac{8 \cdot (-4) + 6 \cdot (5)}{10} = \frac{-2}{5}
\]

\[
\frac{dl}{dt} = -\frac{2}{5}
\]
14. (18 points) Sketch the graph of the function \( f(x) = \frac{x + 2}{(x + 1)^2} \). For this function, \( f'(x) = -\frac{x + 3}{(x + 1)^3} \) and \( f''(x) = \frac{2(x + 4)}{(x + 1)^4} \).

<table>
<thead>
<tr>
<th>Horizontal asymptote(s)</th>
<th>( y = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical asymptote(s)</td>
<td>( x = -1 )</td>
</tr>
<tr>
<td>Increasing</td>
<td>((-3, -1))</td>
</tr>
<tr>
<td>Decreasing</td>
<td>((-\infty, -3) \cup (-1, \infty))</td>
</tr>
<tr>
<td>Concave up</td>
<td>((-4, -1) \cup (-1, \infty))</td>
</tr>
<tr>
<td>Concave down</td>
<td>((-\infty, -4))</td>
</tr>
<tr>
<td>Relative max/min</td>
<td>Relative min ( x = -3 ) no relative max</td>
</tr>
<tr>
<td>Inflections</td>
<td>( x = -4 )</td>
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</tbody>
</table>