1. In each part below give the precise definition in one or more full sentences.
(a) The *span* of a set of vectors $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$;
(b) A *linearly independent* set of vectors $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$;
(c) A *subspace* of $\mathbb{R}^n$;
(d) A *basis* of a subspace W of $\mathbb{R}^n$;
(e) An *eigenvector* and corresponding *eigenvalue* of a square matrix A.
(f) An *eigenspace* of a square matrix A.

2. Suppose that A is an $m \times n$ matrix.
(a) Define the *null space* Null(A) of A.
(b) Show that Null(A) is a subspace of $\mathbb{R}^n$ by checking the conditions in the definition of a subspace.
(c) Define the *column space* Col(A) of A.
(d) Show that Col(A) is a subspace of $\mathbb{R}^m$ by checking the conditions in the definition of a subspace.

3. Find the $A = LU$ factorization (that is, find L and U) of $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 2 & -2 & 3 \\ 0 & 3 & 2 & 7 \end{bmatrix}$. Then use it to solve $A\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 10 \\ 12 \end{bmatrix}$ by solving two equations: one with L and then one with U.

4. The matrix $A = \begin{bmatrix} 3 & 6 & 1 & 0 & 7 \\ 2 & 4 & 0 & 1 & 10 \\ 1 & 2 & 1 & -1 & -3 \\ 0 & 0 & 0 & 3 & 12 \end{bmatrix}$ has reduced row echelon form $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
(a) Use R to determine the dimensions of the spaces Col(A), Null(A), Row(A), and Null(A^T).
(b) Find bases for the spaces Col(A), Null(A), and Row(A). The number of vectors in each basis set should be consistent with the dimensions you found in (a).

5. Classify each statement as true or false and give a brief justification of your answer.
(a) If A is a square matrix and $A\mathbf{x} = \mathbf{0}$ has a unique solution then the equation $A\mathbf{x} = \mathbf{b}$ is always consistent.
(b) The square matrix A is invertible if and only if $\det A = 0$.
(c) If $\mathbf{b}$ is a given nonzero vector, then the set of all solutions $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ is a subspace.
(d) If A is an $m \times n$ matrix and $n > m$ then the nullspace of A is not $\{\mathbf{0}\}$.
(e) If A is an $m \times n$ matrix then $\dim \mathrm{Null}\, A + \dim \mathrm{Row}\, A = n$.
(f) The rank of a matrix A is equal to the nullity of A^T.
(g) If A is an $n \times n$ matrix and rank $A < n$ then 0 is a root of the characteristic polynomial of A.
(h) If λ is an eigenvalue of A with algebraic multiplicity r and W is the corresponding eigenspace then $\dim W$ can take any value from 0 to r.
(i) Every $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

6. In each case below let W be indicated set of vectors. Determine whether W is a subspace of $\mathbb{R}^3$. If it is, give $\dim W$. If $\dim W \geq 1$ find a basis for W.

(a) $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} \right\}$; (b) $\left\{ \begin{bmatrix} r \\ -8s \\ r+s \end{bmatrix} : r, s \in \mathbb{R} \right\}$; (c) $\left\{ \begin{bmatrix} r+s \\ -8(r+s) \\ 2r+2s \end{bmatrix} : r, s \in \mathbb{R} \right\}$;

(d) $\left\{ \begin{bmatrix} r \\ -8s \\ r+s+1 \end{bmatrix} : r, s \in \mathbb{R} \right\}$; (e) $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$; (f) $\left\{ \begin{bmatrix} r \\ -8r \\ 2r \end{bmatrix} : r = 0 \right\}$.

7. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 3 & 4 \\ 1 & -2 & 1 & 2 \\ 3 & -3 & -2 & 1 \end{bmatrix}$.

(a) Evaluate $\det A$ by a cofactor expansion along the first row.
(b) Evaluate $\det A$ by a cofactor expansion along the second row.
(c) Evaluate $\det A$ by row reduction of A to upper triangular form U. (*Don't calculate* $\operatorname{rref}(A)$.)

8. Let $A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$ be a 3×3 matrix with row vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Assume that $\det A = 5$.

(a) Find row operations that transform A into the matrix $B = \begin{bmatrix} \mathbf{c} + 3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix}$. Then calculate $\det B$.

(b) Let $C = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix}$. Find the determinant of the matrix AC^3A^T.

9. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}$.

(b) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

10. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and let $A = \mathbf{v}\mathbf{v}^T$. Note that A is a 3×3 matrix.

(a) Show that $\mathbf{v}$ is an eigenvector of A. What is the eigenvalue? (Hint: compute $A\mathbf{v}$ using the associative property of matrix multiplication.)
(b) Show that $\mathbf{v}$ is a basis for $\operatorname{Col} A$. (Hint: Show that each column of A is a multiple of $\mathbf{v}$.)
(c) What is $\dim \operatorname{Null} A$?
(d) Find the characteristic polynomial of A, the eigenvalues of A, and their algebraic multiplicities.
(e) Show that A is diagonalizable (you don't need to find all the eigenvectors).
(f) If $A = PDP^{-1}$ with D diagonal, what are the diagonal entries of D?

11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial of A and the eigenvalues of A. Give the algebraic multiplicity of each eigenvalue.
(b) For each eigenvalue find a basis for the corresponding eigenspace.
(c) Determine whether or not A is diagonalizable.

12. Do the True-False questions from Sections 2.6, 3.1, 3.2, 4.1–4.3, 5.1–5.3 that are listed in the homework assignments.