Assignment 7

Turn in starred problems Wednesday, March 22, at the beginning of the period. See the remarks below for hints or modifications of several of these problems.

Exercises from Abbott, *Understanding Analysis*:

Section 3.3: 13

Section 3.4: 1, 2, 4, 5*, 6, 7*, 8*, 9(a,b)*

7.A* (a) Suppose that $\{K_1, \ldots, K_n\}$ is a finite collection of open-cover compact sets. Prove, directly from the definition of open-cover compact, that $\bigcup_{k=1}^n K_k$ is open-cover compact.

(b) Suppose that $\{K_\lambda \mid \lambda \in \Lambda\}$ is a collection of open-cover compact sets. Prove, directly from the definition of open-cover compact, that $\bigcap_{\lambda\in\Lambda} K_\lambda$ is open-cover compact.

7.B* (a) Suppose that $A \subset \mathbb{R}$ is a nonempty bounded set such that if $a, b \in A$ and $a < c < b$ then $c \in A$ (compare Theorem 3.4.7). Prove that for some $x, y \in \mathbb{R}$ with $x \leq y$, A is one of (x, y), $[x, y)$, $(x, y]$, and $[x, y]$.

(b) Prove a similar result if A is bounded below but not above.

(c) What would be the (similar) conclusion be if A were given to be bounded above but not below, or unbounded both above and below? In this part you do not have to prove your conclusion.

7.C (Extra credit; turn in in lecture 3/23 if you do it.) In this problem we define, for I an interval, $|I|$ to be the length of the interval, and for A a finite union $\bigcup_{\lambda\in\Lambda} I_\lambda$ of pairwise disjoint intervals, $|A| = \sum_{\lambda\in\Lambda} |I_\lambda|$.

Now modify the construction of the Cantor set as follows: Take $C_0 = [0, 1]$. Assuming inductively that for $n \geq 0$, $C_n = \bigcup_{I\in\mathcal{F}_n} I$, where $\mathcal{F}_n$ is a collection of 2^n pairwise-disjoint closed intervals, define $\mathcal{F}_{n+1}$ to be the collection of intervals obtained by removing, from each interval $I \in \mathcal{F}_n$, a centered open interval of length $\frac{1}{(n+1)^2}|I|$. Finally, define the Cantor-like set C by $C = \bigcap_{n\in\mathbb{N}} I_n$.

(a) Show that C is perfect.

(b) Find $|C| = \lim_{n\to\infty} |C_n|$. (Hint: in this case $|C| > 0$; this is in contrast to the result for the usual Cantor set and for the set in Exercise 3.4.4.)

Comments, hints and instructions:

3.4.9: For (a) you need give only an informal argument (hint: think about length). (c) is not assigned; the first question is vague and the second is, I think, difficult. If you want to try this second part (with a rigorous proof) you can do so for extra credit.