RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2013, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Assume that $x_j : [0,1] \to \mathbb{R}$, $j \geq 1$ is a sequence of Lebesgue measurable functions such that

$$\int_0^1 |x_j(t)|\; dt \leq A < \infty$$

for all $j \geq 1$, where A is independent of j.
Define

$$y_j(s) = \int_0^1 f(s,t) x_j(t)\; dt\ .$$

Prove the following:
(a) Each function $y_j(s)$ is continuous;
(b) There exists a subsequence (j_k), $k \to \infty$ such that $y_{j_k}(s)$ converges, uniformly in s, to a continuous function $y(s)$.

2. Prove that the series

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}$$

converges for every complex number z belonging to the half-plane

$$H = \{z \in \mathbb{C} : Re\ z > 0\}\ ,$$

and that the sum is a holomorphic function on H.
Hint: Study the series

$$\sigma(z) = \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)^z} - \frac{1}{(2k)^z} \right)$$

and estimate the terms of this series by writing each of them as an integral.

3. Let A and B be complex square matrices such that $AB - BA = B$. Prove that $AB - BA$ is not invertible.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Show that for each $n > 1$ the integral

$$I_n = \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} n \sin\left(\frac{x}{n}\right) dx$$

is absolutely convergent and compute the limit $\lim_{n\to\infty} I_n$.
You may assume that

$$\left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{x}{n+1}\right)^{n+1}$$

for all $x \geq 0$ and all positive integers n.

5. Let F be a set of holomorphic functions on an open subset U of the complex plane. We say that F is **bounded** if the number

$$c_{F,K} = \sup\{|f(z)| \ : \ z \in K, f \in F\}$$

is finite for every compact subset K of U. Let

$$F' = \{f' \ : \ f \in F\}.$$

Prove that if F is bounded then F' is bounded.

6. Let G be a finite group and let P be a Sylow p-subgroup of G. Suppose that the number of Sylow p-subgroups of G is at least $|G : P|$, where $|G : P|$ is the index of P in G. Show that for every $g \in G$, g lies in the subgroup generated by P and $g^{-1}Pg$.

7. Construct a biholomorphic map (i.e., a conformal map) from the half-strip

$$S = \{x + iy \in \mathbb{C} : \ 0 < x < 1, y > 0\}$$

onto the unit open disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$.

The exam continues on next page

8. Let f be a complex valued measurable function on the interval $[0,1]$ such that

$$\int_0^1 \int_0^1 |f(x) - f(y)| \, dxdy < \infty.$$

Show that $f \in L^1[0,1]$.

9. Let R be a commutative ring with an identity and without zero divisors. Prove that the ring of polynomials $R[x]$ over R is a principal ideal ring if and only if R is a field.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2013, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let X be a connected topological space and suppose that every point of X has an open neighbourhood which is contractible. Let x_0, x_1 be two different points of X. Show that there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$.

2. Let f and $\{f_k\}$ be functions in $\mathbb{R}^n$ which are Lebesgue measurable and real valued, and suppose that $\{f_k\}$ converges in measure to f on $\mathbb{R}^n$. If there is a function $\varphi \in L^1(R^n)$ such that $|f_k| \leq \varphi$ in R^n for all k, show that

$$\int_{\mathbb{R}^n} f_k \, dx \to \int_{R^n} f \, dx.$$

3. Prove that the center of a group of order p^n, where p is prime and n is a positive integer, contains more than one element.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let E_1 and E_2 be measurable subsets of the unit ball in $\mathbb{R}^n$. Suppose there is a constant c such that for all $x \in E_1$, there is a cube Q_x satisfying $x \in Q_x$ and $|Q_x| \leq c|E_2 \cap Q_x|$. Show that there is a constant C depending only on c and n such that $|E_1| \leq C|E_2|$.

5. Let $1 < p \leq \infty$. Let f and $\{f_k\}$ be functions in $L^p[0,1]$ such that $f_k \to f$ a.e. in $[0,1]$. If there exists a finite constant M such that $||f_k||_p \leq M$ for all k, show that for every $g \in L^{p'}[0,1]$, where $1/p + 1/p' = 1$,

$$\int_0^1 f_k g \, dx \to \int_0^1 fg \, dx.$$

6. Suppose that G is a group with a chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{n-1} \supseteq G_n = \{e\}$$

such that for all $i = 1, \ldots, n$, G_i is a normal subgroup of G and G_{i-1}/G_i is cyclic. Suppose that N is a nonidentity normal subgroup of G such that no subgroup of N other than N itself and the trivial subgroup $\{e\}$ is normal in G. Prove that N has prime order.

7. Let A be a linear operator in an n-dimensional complex vector space E. Prove that the set of of linear operators in E commuting with A is a vector space of dimension $\geq n$.

The exam continues on next page

8. Use residues and contour integrals to evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx.$$

9. Let f be a holomorphic function on the disc $D_R = \{z \in \mathbb{C} : |z| < R\}$ such that $|f(z)| \leq A$ for all $z \in D_R$ and $f(0) = f'(0) = 0$. Prove that

$$\left|f\left(\frac{R}{2}\right)\right| \leq \frac{A}{4}.$$

Exam Day 2 End