RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 14, 2010, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Consider the integral

$$I(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Prove that the integral converges for $z \in U$, where $U = \{z \in \mathbb{C} : \text{ Re } z > 0\}$, and that the function defined by the integral is analytic on U.

2. Let λ be Lebesgue measure on $(\mathbb{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Recall that λ is a complete measure.
Show that the product measure $\lambda \times \lambda$ on the product σ-algebra $\mathcal{L} \otimes \mathcal{L}$ is not a complete measure. (HINT: Show first that if $A \in \mathcal{L}$ but $B \notin \mathcal{L}$, then $A \times B \notin \mathcal{L} \otimes \mathcal{L}$.)

3. Prove that a matrix is nilpotent if and only if all its eigenvalues are equal to zero.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Compute the following integral by using the residue theorem:

$$\int_0^\infty \frac{\sin x}{x} dx.$$

(Justify your computation and, in particular, if you take the limit of the integrals over some variable contour or part of a contour, explain why this passage to the limit is justified.)

5. Assume that U is an open subset of the complex plane $\mathbb{C}$, and $f : U \to \mathbb{C}$ is a holomorphic function. Assume, in addition, that f is one-to-one, that is, $f(z) \neq f(z')$ whenever $z \neq z'$. Prove that the derivative $f'(z)$ is different from zero for every $z \in U$.

6. i. State the Fubini-Tonelli theorem(s).

ii. Use polar coordinates and a change of variables formula to evaluate

$$\int_{\mathbb{R}^2} e^{-|x|^2} d\lambda^2,$$

where λ^2 denotes two-dimensional Lebesgue measure.

iii. Use the result of Part **ii** to evaluate

$$\int_{\mathbb{R}^n} e^{-|x|^2} d\lambda^n,$$

where λ^n denotes n-dimensional Lebesgue measure.

In the calculations of Parts **ii** and **iii**, cite all uses of Fubini-Tonelli.

7. Consider the Hilbert space $H = L^2(\mathbb{R}, \lambda)$ where λ denotes Lebesgue measure on $\mathbb{R}$. Recall that a sequence $\{u_k\}_{k\in\mathbb{N}}$ in H is said to *converge weakly* to $u \in H$ if for all $v \in H$ you have $\int \overline{v}(u_k - u)d\lambda \to 0$ as $k \to \infty$; we say that $\{u_k\}_{k\in\mathbb{N}}$ *converges strongly* to $u \in H$ if $\int |u - u_k|^2 d\lambda \to 0$ as $k \to \infty$.

Suppose that a sequence $\{u_k\}_{k\in\mathbb{N}}$ in H converges weakly to $u \in H$ and that $\|u_k\|_2 \to \|u\|_2$ as $k \to \infty$. Show that $\{u_k\}_{k\in\mathbb{N}}$ converges strongly to u in H.

8. Prove that every group G of order 45 is abelian. **Hint**: You may try to prove first that any group of order p^2, where p is a prime number, is abelian.

9. Let R denote the ring of upper triangular $n \times n$-matrices with coefficients in the integers $\mathbb{Z}$. Find all ideals in R.

Recall that an *ideal* I in R (or any ring) is an additive subgroup such that for every $r \in R$ and $x \in I$ both rx and xr are in I.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 15, 2010, Day 2

This examination is given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let A and B be normal subgroups of a group G such that $A \cap B$ is the identity subgroup. Prove that $xy = yx$ for any $x \in A$, $y \in B$.

2. State the monotone convergence theorem and Fatou's lemma and then prove the latter with the help of the former. Also give an example showing that the conclusion of Fatou's lemma can be false if its key hypothesis is violated.

3. Recall that, if $L \in \mathbb{R}$ is a positive constant, then an *L-Lipschitz map* from a metric space (A, d_A) to a metric space (B, d_B) is a map $\varphi : A \to B$ such that $d_B(f(x), f(x')) \leq L \cdot d_A(x, x')$ for all $x, x' \in A$.

Let (X, d_X) and (Y, d_Y) be complete metric spaces, let S be a dense subset of X, and let $f : S \to Y$ be an L-Lipschitz map. Show that there is a unique L-Lipschitz map $\hat{f} : X \to Y$ such that $\hat{f}|S = f$ (that is $\hat{f}$ is an extension of f).

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $\mathbb{D}$ denote the open unit disc, $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. Find a biholomorphic (that is, holomorphic with holomorphic inverse) map from the open set

$$U = \mathbb{D} - \{x + iy : \ y = 0 \text{ and } -1 \leq x \leq 0\}$$

(that is, $\mathbb{D}$ with the segment $\{x + iy : \ y = 0 \text{ and } -1 \leq x \leq 0\}$ removed) onto $\mathbb{D}$.

5. Explain why the Banach-Tarski paradox shows that there can be no finitely additive, rotation- and translation-invariant measure on all subsets of $\mathbb{R}^3$.

6. Give an example each of a function which is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$, respectively in $L^3(\mathbb{R})$ but not in $L^2(\mathbb{R})$. Prove that any function which is in $L^1(\mathbb{R})$ and in $L^3(\mathbb{R})$ is also in $L^p(\mathbb{R})$ for all $p \in (1, 3)$. You may assume Hölder's inequality.

7. Prove that any monic polynomial $f(x)$ over an algebraically closed field is the minimal polynomial of some square matrix A.

8. Let R be a commutative ring with unit, such that $xy = 0$ implies either $x = 0$ or $y = 0$ for all x, y in R.

Prove that $R[x]$ is a principal ideal ring if and only if R is a field.

9. Let (X, d) be a metric space, and let $K(X)$ denote the collection of all nonempty compact subsets of X. Define a function, $d_h : K(X) \times K(X) \to \mathbb{R}$ by letting $d_h(A, B) = \inf\{\varepsilon : A \subseteq U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}$, where $U_\varepsilon(S) = \{x \in X : \ d(x, S) < \varepsilon\}$, and the distance $d(x, S)$ from a point $x \in X$ to a nonempty subset S of X is defined by $d(x, S) = \inf\{d(x, s), \ s \in S\}$.

Prove that $(K(X), d_h)$ is a complete metric space.

Exam Day 2 End