Rutgers University - Graduate Program in Mathematics

Written Qualifying Examination

Spring 1998

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

Typeset by $\mathcal{AMS}$-TEX

First Day – Part I: Answer each of the following three questions.

1. Let A be an $n \times n$-matrix over an algebraically closed field k. Let $P(x)$ be a polynomial over k. Show that $\lambda \in k$ is an eigenvalue of $P(A)$ if and only if $\lambda = P(\gamma)$ where γ is an eigenvalue of A.

2. Let D be an open connected domain in the complex plane $\mathbb{C}$ and $f(z)$ an analytic function defined on D. Assume that there is a smooth curve Γ in $\mathbb{C}$ such that $f(D) \subseteq \Gamma$. Prove that f is constant.

3. Let f be a real valued function on an interval $I \subseteq \mathbb{R}$ such that $f'(x)$ exists for all $x \in I$. Show, that f' has the intermediate value property, i.e., for all $a, b \in I$ and $c \in \mathbb{R}$ with $f'(a) < c < f'(b)$ there is ξ in the interval bounded by a and b such that $f'(\xi) = c$.

First Day – Part II: Answer <u>three</u> out of the following six questions.

4. Show that there is no simple group of order 224.

5. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $0 < m(E) < \infty$. Let $\chi_{E}(x)$ denote the characteristic function of E.

a) Prove that

$$\varphi(x) = \int_{\mathbb{R}} \chi_{E}(y)\chi_{E}(x+y)dy.$$

is a continuous function on $\mathbb{R}$. [*Hint:* Observe that $\varphi(x)$ is an L^2-scalar product.]

b) Let $F = \{x - y \mid x, y \in E\}$. Prove that there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq F$.

6. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx$$

for $a > 0$. Hint: $\cos x = Re(e^{ix})$.

7. Let X be a metric space. A point $x \in X$ is called isolated if the set $\{x\}$ is open. Show that the following two properties are equivalent:

a) Every subset of X is open or closed.

b) X contains at most one non-isolated point.

8. Let f be an analytic mapping of the open unit disk D into itself. Suppose there are two points $z_1 \neq z_2 \in D$ with $f(z_1) = z_1$ and $f(z_2) = z_2$. Show that f is the identity map.

9. Let A and B be real symmetric $n \times n$-matrices. Assume that A is positive definite. Show that there is an invertible real $n \times n$-matrix S such that both SAS^t and SBS^t are diagonal. (Here S^t is the transpose of S.)

Second Day – Part I: Answer each of the following three questions.

1. Let $f(x)$ be a continuous function on $\mathbb{R}$ with compact support and $\alpha > 1$. Show that

$$g_N(x) = \sum_{n=0}^{N} \frac{f(x-n)}{(n+1)^\alpha}$$

converges in L^2 as $N \to \infty$.

2. For N a positive integer, determine

$$\int_{|z|=\pi(N+\frac{1}{2})} e^{iz} cot(z)\, dz.$$

3. Let A be the ring of C^∞-functions on $\mathbb{R}$.

a) Show that

$$P := \{f \in A \mid \frac{d^k f}{dx^k}(0) = 0 \text{ for all } k \geq 0\}$$

is a prime ideal of A.

b) Show that

$$Q := \{f \in A \mid \text{there is a neighborhood } U \text{ of } 0 \text{ with } f|_U \equiv 0\}$$

is an ideal but not a prime ideal of A.

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let P be the group of all permutations of $\mathbb{Z}$. For $\varphi \in P$ define its *support* as $Supp(\varphi) := \{x \in \mathbb{Z} \mid \varphi(x) \neq x\}$. Let A_∞ be the set of all $\varphi \in P$ with $S := Supp(\varphi)$ finite and such that $\varphi|_S$ is an even permutation of S.

a) Show that A_∞ is a subgroup of P.
b) Show that A_∞ is a simple group.

5. A sequence of L^2-functions $\{f_n\}$ on $\mathbb{R}$ is said to converge weakly to $f \in L^2$ if for every $g \in L^2$

$$\int f_n g \longrightarrow \int fg \quad \text{for } n \to \infty.$$

Give an example (with proof) of a sequence of functions which converges weakly to 0 but does not converge to 0 in the L^2-norm.

6. Let S_n be the symmetric group on n letters.

a) Show: an automorphism of S_n is inner if and only if it maps transpositions to transpositions.
b) Show (assuming part a): every automorphism of S_7 is inner.

7. Let f and g be functions which are holomorphic in a neighborhood of the closed unit disk $|z| \leq 1$. Assume $|g(z)| < |f(z)|$ for all z with $|z| = 1$. Prove that f and $f + g$ have the same number of zeroes (counted with multiplicities) in the closed unit disk. [Hint: consider $f + tg$ for $0 \leq t \leq 1$.]

8. Let $f(z)$ be an analytic function on $\mathbb{C}$ such that there exists a constant $C > 0$ and an integer $k \geq 0$ with

$$|f(z)| \leq C(1 + |z|)^k$$

for all $z \in \mathbb{C}$. Prove that f is a polynomial in z.

9. Let f be a non-negative measurable function on the interval $I = [0, 1]$ such that $\int_I fg < \infty$ for every non-negative measurable function g on I with $\int_I g < \infty$. Show that there is a constant $M > 0$ with $f < M$ almost everywhere.