RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Jan 2018

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that any homomorphism from a finitely generated abelian group *onto* itself is an automorphism.

2. Let K be a field. Let $K[[x]]$ denote the ring of formal power series, whose elements are expressions of the form $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in K$ and the usual addition and multiplication. Find, with proof, all ideals of $K[[x]]$.

3. **(A)** Prove that for any square matrices A and B of size n with coefficients in some field the characteristic polynomial of AB equals that of BA.

(B) Give an example of square matrices A and B such that the minimal polynomial of AB does not equal that of BA.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. **(A)** Prove that a Sylow 2-subgroup of the symmetric group S_4 is isomorphic to the dihedral group D_4 of 8 elements.

(B) Prove that a Sylow p-subgroup of the symmetric group S_n is non-abelian if and only if $n \geq p^2$.

5. Let I be a maximal ideal of $\mathbb{Z}[x]$. Prove that $\mathbb{Z}[x]/I$ is a finite field.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let $\lambda > 1$ be a real number. Show that the equation $ze^{\lambda - z} = 1$ has a real solution in the unit disk, and that there are no other solutions in the unit disk.

2. Let $\gamma(t) : [0, b] \to \mathbb{C}$ be a piecewise smooth function describing a curve Γ in the complex plane.

(A) For $a \notin \Gamma$ let

$$h(u) = \int_0^u \frac{\gamma'(t)}{(\gamma(t) - a)}\, dt.$$

Differentiate $e^{-h(u)}(\gamma(u) - a)$ and prove that $e^{h(u)} = (\gamma(u) - a)/(\gamma(0) - a)$ for all $0 \leq u \leq b$.

(B) Use **(A)** to show that if Γ is a closed path then $\int_\Gamma (z - a)^{-1} dz$ is an integer multiple of $2\pi i$. Show that this integral is zero if Γ is contained in the interior of a disk not containing a.

3. For real $b > 1$ let Ω_b be the open set in the complex plane defined by

$$\Omega_b := \{z \in \mathbb{C} : \ |z - bi| > 1, \, \mathrm{Im}(z) > 0\}.$$

Find some $d = d(b)$ such that Ω_b is biholomorphic to the annulus

$$\{z \in \mathbb{C} : d < |z| < 1\}.$$

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Define $D := \{z \in \mathbb{C}, \ 2 < |z| < 3\}$. Let f be a holomorphic function over D that is continuous over $\overline{D}$.

(A) Suppose that $\max_{|z|=2} |f(z)| \leq 2$ and $\max_{|z|=3} |f(z)| \leq 3$. Prove that $|f(z)| \leq |z|$ on D.

(B) Suppose that $|f(z)| = |z|$ for $|z| = 2$ and $|z| = 3$. Suppose furthermore that $f(z)$ does not have any zeros in D. Prove that $f(z) = e^{\imath\theta} z$ for some constant $\theta \in [0, 2\pi]$.

5. **(A)** Suppose that a is a complex number and f is a holomorphic function in the disk centered at a with radius $r > 0$. Let $C_{a,\epsilon}$ be the semi-circle centered at a with radius ϵ clockwise from $a - \epsilon$ to $a + \epsilon$. Here we assume that $0 < \epsilon < r$. Prove that

$$\lim_{\epsilon \to 0} \int_{C_{a,\epsilon}} \frac{f(z)}{z - a} dz = -\pi i f(a).$$

(B) Compute by using the residue theorem or Cauchy formula the following integral:

$$\int_0^\infty \frac{\ln x}{x^2 - 1} dx.$$

Hint: Consider a contour composed of a semicircle with center zero and radius R, completed by a portion of the real axis having small half-circular indentations at ± 1 and 0.

End of Session 2

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GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Jan 2018

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let S be a connected metric space, with metric $d : S \times S \to \mathbb{R}_+$. Let $q \in S$. Show that if $S\backslash\{q\}$ is non-empty, then it is not compact.

2. Let m be the Lebesgue measure. For any $A \subseteq \mathbb{R}$, define

$$m^*(A) = \inf\{\sum_{i=1}^{\infty} m(I_i);\ \{I_i\} \text{ is a cover of } A \text{ by open intervals}\Big\}.$$

(A) Show that m^* is an outer measure on the subsets of $\mathbb{R}$.

(B) Let A and B be subsets of $\mathbb{R}$ and assume that

$$d(A, B) := \inf\{|x - y| : x \in A, y \in B\} > 0.$$

Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

3. Let $S = [0, 1] \times [0, 1] = [0, 1]^2$, let λ denote the Lebesgue measure on the Lebesgue σ algebra $\Lambda([0, 1])$, and let λ^2 denote the product measure on the product σ algebra $\Lambda \otimes \Lambda$. Let

$$f(x, y) = \frac{\cos(10x + 17y)}{1 - xy}.$$

Determine, with proof, whether the following integrals $I(f)$, $J(f)$ and $K(f)$ are finite, and if so, whether there holds equality between them:

$$I(f) = \int_{[0,1]} \Big(\int_{[0,1]} f(x, y)\lambda(dy)\Big)\lambda(dx),\ \ J(f) = \int_{[0,1]} \Big(\int_{[0,1]} f(x, y)\lambda(dx)\Big)\lambda(dy),$$

$$K(f) = \int_S f(x, y)d\lambda^2.$$

To get full credit, you must explicitly formulate the theorem(s) you are using.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let λ denote Lebesgue measure on $\mathbb{R}$. Let $x \in \mathbb{R}$, and let $f : x \mapsto f(x) \in \mathbb{R}$ be Lebesgue measurable. For Borel sets $B \subset \mathbb{R}$, define

$$\mu(B) = \lambda(\{x : f(x) \in B\}).$$

Show that μ is a measure, and that

$$\int_{\mathbb{R}} g(y) d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x) d\lambda(x)$$

for all g such that the integrals make sense.

5. Let $f : [0, \infty) \to \mathbb{R}$ be a Borel measurable function such that $|f(x)| \leq 1 + x^2$ for all $x \geq 0$ and such that f is continuous at 0. Prove that

$$\lim_{n\to\infty} \int_0^n \frac{f(x/n)}{(1+x)^4}\, dx$$

exists and calculate the limit.

End of Session 3