RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2018

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Give an example of an integral domain R and an ideal I in R such that *all* of the following statements hold. The ideal I is not principal, it is not maximal, and it is prime.

2. Let p and q be distinct primes. Let $\bar{q} \in \mathbb{Z}/p\mathbb{Z}$ denote the class of q modulo p and let k denote the order of $\bar{q}$ as an element of $(\mathbb{Z}/p\mathbb{Z})^*$. Prove that no group of order pq^l with $1 \leqslant l \leqslant k$ is simple.

3. Let M be a square matrix with complex coefficients. We consider the usual matrix exponential

$$\exp(M) = \sum_{j=0}^{\infty} \frac{1}{j!} M^j.$$

Prove that $\exp(M)$ is equal to the identity matrix if and only if M is diagonalizable with eigenvalues in $2\pi \mathrm{i}\, \mathbb{Z}$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $G = \mathbb{Q}/\mathbb{Z}$ be the quotient of the additive group of rational numbers by the subgroup of integers.

(A) Prove that every finitely generated subgroup of G is a finite cyclic group.

(B) Prove that G is not isomorphic to $G \oplus G$ as an abelian group.

5. Let G be a finite subgroup of the group of real $n \times n$ matrices with nonzero determinant such that all elements of G are symmetric matrices. Prove that G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $k \geqslant 0$.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. In this problem, U is a connected domain in $\mathbb{C}$.

(A) Let $h(z)$ be a harmonic function in U, and $f : U \mapsto U$ be a holomorphic function. Prove that $h \circ f(z)$ is a harmonic function in U.

(B) Let $h(z)$ be a *real valued* harmonic function in U such that $(h(z))^2$ is also a harmonic function in U. Prove that $h(z)$ must be a constant.

2. Let $z_1 \neq z_2 \in \mathbb{C}$.

(A) Construct all biholomorphic maps of the complex plane which have z_1 and z_2 as their fixed points; make sure to justify your construction.

(B) Construct all biholomorphic maps of the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ which have z_1 and z_2 as their fixed points.

3. Use the calculus of the residues to evaluate the integral $\int_0^\infty \frac{1}{1+x^3} dx$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. (A) Suppose that U is a bounded domain in the complex plane, that f is holomorphic in U and continuous over $\overline{U}$, and that $|f(z)| = 1$ for all $z \in \partial U$. Prove that, unless f is a constant function, $f(z) = 0$ must have a solution in U.

(B) Suppose that f is an entire function, and that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that $f(z) = az^n$ for some $a \in \mathbb{C}$ with $|a| = 1$ and $n \in \mathbb{Z}_{\geqslant 0}$. (*Hint: Note that the family $\phi_c(z) := (z-c)/(1-\bar{c}z)$, for $|z| < 1$, is meromorphic on $\mathbb{C}$, satisfies $|\phi_c(z)| = 1$ for $|z| = 1$, and $\phi_c(1/\bar{z})\overline{\phi_c(z)} = 1$. You may want to study $f(z)$ in relation to this family.*)

5. Suppose that $\{w_n(z)\}$ is a sequence of holomorphic functions in a domain U of the complex plane such that

(i). each w_n is a solution in U to the differential equation $w''(z) = F(w'(z), w(z), z)$, where F is continuous in its arguments; and (ii). there exists a function $w_\infty(z)$ on U such that $w_n(z) \to w_\infty(z)$ pointwise in U, and that on any compact subdomain of U, the family $\{w_n(z)\}$ is uniformly bounded.

Prove that $w_\infty(z)$ is also a solution in U to the differential equation $w''(z) = F(w'(z), w(z), z)$.

End of Session 2

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2018

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. **Write the definition** of a separable metric space, then prove the following three statements:

   (a) **Show** that every infinite compact metric space (K, d) is separable.

   (b) Consider the metric space (X, d), where X is the set of bounded sequences of real numbers, i.e of $\mathbf{x} = \{x_n\}_1^\infty$, $x_n \in \mathbb{R}$, equipped with the distance
   $$d(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$
   (this space is known as ℓ^∞). **Show** that (X, d) is *not* separable.

   (c) **Show** that if (X, d) is a separable metric space then the cardinality of X can *not* be larger than the cardinality of $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ (or in other words the cardinality of $\mathbb{R}$).

2. Let f be an integrable real-valued function on $\mathbb{R}$. Define a function $g : \mathbb{R} \mapsto \mathbb{R}$ by letting
   $$g(x) = \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2} dy\,.$$

   (a) **Prove** that g is continuous.

   (b) **Prove** that g is continuously differentiable.*

   (* Help for **2(b)**: State a theorem **without proof** which lists suitable conditions that allow you to "differentiate under the integral sign," then verify these conditions for this problem.)

3. Let $\pi(x, y) = x$ denote the projection of $\mathbb{R}^2$ onto $\mathbb{R}$, and let $\pi(A)$ denote the image under π of a subset A of $\mathbb{R}^2$.

   (a) Let μ^* be an outer measure on the subsets of $\mathbb{R}$. Show that $\nu^*(A) := \mu^*(\pi(A))$ is an outer measure on the subsets of $\mathbb{R}^2$.

   (b) Let λ^* be Lebesgue outer measure on the subsets of $\mathbb{R}$, and let $\rho^*(A) = \lambda^*(\pi(A))$. Show that if $A = B \times \mathbb{R}$, where B is a Lebesgue measurable subset of $\mathbb{R}$, then A is a ρ^* measurable set. Show where the assumption that A has this particular form is used.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $\Omega := [0, 1]$ with Lebesgue measure.

(a) Let $f_n(x) = \cos 2\pi n x$. **Show** that $f_n \to 0$ weakly in $L^2(\Omega)$, but f_n does not converge to 0 a.e. in Ω or in measure.

(b) Let $f_n(x) = n\chi_{(0,\,1/n)}$. **Show** that $f_n \to 0$ a.e. and in measure, but f_n does not converge to 0 weakly in $L^p(\Omega)$ for any $p \geqslant 1$.

Recall: *A sequence of* $\{g_n\}$ *of* L^p*-functions converges to* g *weakly in* $L^p(\Omega)$ *if* $\int_\Omega g_n\varphi \to \int_\Omega g\varphi$ *for every* $\varphi \in L^q(\Omega)$ *(the dual of* $L^p(\Omega)$*) with* $1/p + 1/q = 1$.

5. Let f be an integrable real-valued function on $\mathbb{R}$. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \mapsto \mathbb{R}$ be a nonnegative measurable function. The hypograph of f is the subset $HG(f)$ of $X \times \mathbb{R}$ defined by

$$HG(f) = \Big\{ (x,t) \in X \times \mathbb{R} : 0 \leqslant t \leqslant f(x) \Big\}.$$

(That is, $HG(f)$ is the "region under the graph of f".)

(a) **Prove** that the set $HG(f)$ is $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$-measurable, where $\mathcal{B}_{\mathbb{R}}$ is the σ-algebra of Borel subsets of $\mathbb{R}$, and $\mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$ is the product of the σ-algebras $\mathcal{A}$ and $\mathcal{B}_{\mathbb{R}}$, i.e., the σ-algebra of subsets of $X \times \mathbb{R}$ generated by the products $A \times B$, $A \in \mathcal{A}$, $B \in \mathcal{B}_{\mathbb{R}}$.

(b) Let $h_f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the function given by

$$h_f(t) = \mu\Big(\{x \in X : f(x) \geqslant t\} \Big),$$

where $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$. **Prove** that h_f is Borel measurable and that

$$\int_X f\, d\mu = \int_0^\infty h_f(t)dt\,.$$

End of Session 3