Study Guide for the February 28 Exam, MATH251


The topics regarding the epsilon-delta definition for limits, computing error bounds using the linear approximation and the concept of "differentials" will not be tested on the exam.

Disclaimer: the following guide contains a summary of the contents covered for this midterm. However, you should read the book since anything included there can be evaluated on the exam.

Things you need to know for the exam

Section 13.1

- The vector function \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \) traces out a curve as \( t \) varies.

- We say that \( \mathbf{r}(t) \) has a derivative at \( t \) if \( x(t), y(t), z(t) \) have derivatives at \( t \), and the derivative is

\[
\mathbf{r}'(t) = \frac{d}{dt} \mathbf{r} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}
\]

- The derivative \( \mathbf{r}'(t_0) \) can be interpreted as a vector tangent to the curve at the point \( \mathbf{r}(t_0) \).

- When one thinks of \( \mathbf{r}(t) \) as the position vector of a particle, then \( \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \) is the velocity vector of the particle.

- The speed of the particle is the norm of the velocity vector, that is, \( |\mathbf{v}(t)| \).

- The acceleration of the particle is \( \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \).

- The differentiation rules for vector functions can be found on page 768 of the book.
The most important ones include

\[
\begin{align*}
\frac{d}{dt}(f(t)u(t)) &= f'(t)u(t) + f(t)u'(t) \\
\frac{d}{dt}(u(t) \cdot v(t)) &= u'(t) \cdot v(t) + u(t) \cdot v'(t) \\
\frac{d}{dt}(u(t) \times v(t)) &= u'(t) \times v(t) + u(t) \times v'(t) \\
\frac{d}{dt}(u(f(t))) &= f'(t)u'(f(t))
\end{align*}
\]

→ If \( \mathbf{r}(t) \) has constant length, then \( \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0 \).

Section 13.2

→ The indefinite integral of \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \) is obtained by integrating the individual functions \( x(t), y(t), z(t) \), that is,

\[
\int \mathbf{r}(t)dt = \left( \int x(t)dt \right)\mathbf{i} + \left( \int y(t)dt \right)\mathbf{j} + \left( \int z(t)dt \right)k
\]

→ Likewise, the definite integral of \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \) from \( t = a \) to \( t = b \) is obtained by integrating the individual functions \( x(t), y(t), z(t) \), that is,

\[
\int_a^b \mathbf{r}(t)dt = \left( \int_a^b x(t)dt \right)\mathbf{i} + \left( \int_a^b y(t)dt \right)\mathbf{j} + \left( \int_a^b z(t)dt \right)k
\]

→ You should know how to obtain the trajectory of a particle under ideal projectile motion starting from the differential equation

\[
\begin{align*}
\frac{d^2\mathbf{r}}{dt^2} &= -g\mathbf{j} \\
\mathbf{r} &= \mathbf{r}_0, \frac{d\mathbf{r}}{dt} = \mathbf{v}_0, \quad t = 0
\end{align*}
\]

This is discussed on pages 774 and 775 of the book, and the end result is that

\[
\mathbf{r} = (v_0 \cos \alpha)i + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)j
\]

Section 13.3

→ The length of a smooth curve \( \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \) for \( a \leq t \leq b \) is

\[
L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt
\]

→ The formula for the arc-length parameter \( s(t) \) is exactly the same as the formula for the length \( L \), the only difference being that now \( b \) is allowed to be a variable \( t \)

\[
s(t) = \int_{t_0}^t |\mathbf{v}(\tau)|d\tau
\]
You should be able to find arc-length parameterization of a curve [see Example 2].

The speed of a curve can be computed as the time derivative of the arc-length:

\[ |\mathbf{v}(t)| = \frac{ds}{dt} \]

Notice that if \( t \) happens to be the arc-length parameter, that is, \( t = s \), then \( \frac{ds}{dt} = \frac{ds}{ds} = 1 \), which means that the speed of the particle is one. In other words, under arc-length parametrization a particle moves a unit-speed.

More generally, if a particle moves with speed \( \mathbf{v} \), then the unit tangent vector is

\[ \mathbf{T} = \frac{\mathbf{v}}{||\mathbf{v}||} \]

By parameterizing a curve with the arc-length parameter, one can see that \( \mathbf{T} \) also equals \( \frac{d\mathbf{r}}{ds} \).

**Section 14.1**

A point \((x_0, y_0)\) in a region \( R \) in the \( xy \) plane is an **interior point** of \( R \) if it is the center of a disk of positive radius that lies entirely in \( R \). A point \((x_0, y_0)\) is a **boundary point** of \( R \) if every disk centered at \((x_0, y_0)\) contains points that lie outside of \( R \) as well as points that lie in \( R \).

A region is **open** if every point is an interior point. A region is **closed** if it contains all its boundary points.

A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

The domain of a function \( z = f(x, y) \) of two variables \( x, y \) corresponds to a region on the \( xy \) plane [see example 2].

The graph of \( z = f(x, y) \) is a surface in 3d-space (\( xyz \) space). The intersection of this surface with a plane \( z = c \) of height \( c \), typically gives a curve called the **level curve** of \( f \).

By analogy, for a function \( w = f(x, y, z) \), the points satisfying \( f(x, y, z) = c \) for some constant \( c \) for a **level surface** of \( f \).

**Section 14.2**

We say that \( \lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \) if, for every number \( \epsilon > 0 \), there exists a corresponding \( \delta > 0 \) such that for all \((x, y)\) in the domain of \( f \), \( |f(x, y) - L| < \epsilon \) whenever \( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \).

You will not need to know the \( \epsilon - \delta \) definition for the midterm. Instead, you will need know how to show that a limit exists by arguments similar to the ones you used in previous Calculus courses, for example: using the squeeze theorem, or writing the limit in polar coordinates. Alternatively, you may need to show that a limit
does not exist by considering different paths which approach the point \((x_0, y_0)\), and showing that the limiting value \(f(x, y)\) takes along different paths gives different answers. Example 6 illustrates this technique.

A function \(f(x, y)\) is continuous at the point \((x_0, y_0)\) if: \(f\) is defined at \((x_0, y_0)\), \(\lim_{(x,y)\to(x_0,y_0)} f(x, y)\) exists, and \(\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)\).

### Section 14.3

The partial derivative of \(f(x, y)\) with respect to \(x\) at the point \((x_0, y_0)\) is

\[
\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}
\]

with an analogous definition for \(\frac{\partial f}{\partial y}\). Geometrically, it can be interpreted as the slope of the tangent line to the curve obtained from the intersection of the graph of \(f(x, y)\) with the plane \(y = y_0\) [Figure 14.16]. You need to be familiarized with the different notations for partial derivatives, like \(\frac{\partial f}{\partial x}\) and \(f_x\).

If \(f(x, y)\) and its partial derivatives \(f_x, f_y, f_{xy}, f_{yx}\) are defined throughout an open region containing a point \((a, b)\) and are all continuous at \((a, b)\), then \(f_{xy}(a, b) = f_{yx}(a, b)\), in other words, you can compute mixed partial derivatives in any order that you want.

If the partial derivatives \(f_x, f_y\) of a function \(f(x, y)\) are continuous throughout an open region \(R\), then \(f\) is differentiable at every point of \(R\). The precise definition of differentiability is given on page 831, though you do not need to know it for the exam. Intuitively, it means that the graph of the function is well approximated by its tangent plane. More precisely, if you start at a point \((x_0, y_0)\) and move a small amount \(\Delta x, \Delta y\) so that \((x, y) = (x_0 + \Delta x, y_0 + \Delta y)\), then the difference

\[
\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)
\]

is well approximated by the quantity

\[
f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y
\]

The full statement is given in Theorem 3.

### Section 14.4

The chain rules states that if \(w = f(x, y)\) is differentiable and \(x = x(t), y = y(t)\) are differentiable functions of \(t\), then \(w = f(x(t), y(t))\) is differentiable as a function of \(t\) and

\[
\frac{dw}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]
See page 836 for a diagrammatic presentation of the rule. When \( w = f(x, y, z) \) and \( x, y, z \) are functions of \( t \), a similar differentiation rule applies and we get

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

If \( w = f(x, y, z) \), \( x = g(r, s) \), \( y = h(r, s) \) and \( z = k(r, s) \), and all four functions are differentiable, then \( w \) has partial derivatives with respect to \( r, s \) and

\[
\begin{align*}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\end{align*}
\]

Figure 14.22 gives a graphical interpretation of the previous chain rules.

If \( F(x, y) \) is differentiable and \( F(x, y) = 0 \) defines \( y \) as a function of \( x \), then whenever \( F_y \neq 0 \),

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}
\]

For an equation of the form \( F(x, y, z) = 0 \), if \( F_z \neq 0 \), we have

\[
\begin{align*}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z}
\end{align*}
\]

Section 14.5

The derivative of \( f \) at \( P_0 = (x_0, y_0) \) in the direction of the unit vector \( u = u_1 \mathbf{i} + u_2 \mathbf{j} \) is the number

\[
D_u f(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}
\]

The directional derivative can be computed as

\[
D_u f(P_0) = \nabla f(P_0) \cdot u
\]

where \( \nabla f \) represents the gradient of \( f \):

\[
\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}
\]

For a function \( f(x, y, z) \) of three variables, the directional derivative can be defined in a similar way, and it is still true that

\[
D_u f(P_0) = \nabla f(P_0) \cdot u
\]

but now

\[
\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}
\]

and \( u = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \) is still a unit vector.
Interpretation of the gradient:

- a) At each point \( P \) on the domain of \( f \), \( f \) increases most rapidly in the direction of the gradient vector \( \nabla f \) at \( P \). In fact, if \( u \) is a unit vector parallel to \( \nabla f \), then \( D_u f = |\nabla f| \).

- Similarly, \( f \) decreases most rapidly in the direction of \( -\nabla f \). In fact, if \( u \) is a unit vector in the direction of \( -\nabla f \), then \( D_u f = -|\nabla f| \).

- Finally, if \( u \) is a unit vector orthogonal to \( \nabla f \), then \( D_u f = 0 \). In particular, this means that if one moves along a level curve of \( f \), then the gradient is perpendicular to the tangent vector of the level curve [see Figure 14.31]. Moreover, the equation of the tangent line to a level curve can be written as

\[
fx(x_0,y_0)(x-x_0) + fy(x_0,y_0)(y-y_0) = 0
\]

- The rules for the gradient can be found on page 850.

Chain Rule for Paths: \( \frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot \frac{dr}{dt} \)

Section 14.6

- The equation of the tangent plane to \( f(x,y,z) = c \) at \( P_0 = (x_0,y_0,z_0) \) is

\[
fx(P_0)(x-x_0) + fy(P_0)(y-y_0) + fz(P_0)(z-z_0) = 0
\]

and the equation of the normal line is

\[
\begin{align*}
x &= x_0 + fx(P_0)t \\
y &= y_0 + fy(P_0)t \\
z &= z_0 + fz(P_0)t
\end{align*}
\]

- The tangent plane to the surface \( z = f(x,y) \) of a differentiable function \( f \) at the point \( P_0 = (x_0,y_0,z_0) = (x_0,y_0,f(x_0,y_0)) \) is

\[
fx(x_0,y_0)(x-x_0) + fy(x_0,y_0)(y-y_0) - (z-z_0) = 0
\]

- The linearization of a function \( f(x,y) \) at a point \( (x_0,y_0) \) where \( f \) is differentiable is the function

\[
L(x,y) = f(x_0,y_0) + fx(x_0,y_0)(x-x_0) + fy(x_0,y_0)(y-y_0)
\]

and the linear approximation of \( f \) at \( (x_0,y_0) \) consists in calculating \( f(x_0,y_0) \) as \( L(x_0,y_0) \).