Solutions to review problems for Exam 2 in Math 151

(1) Assume $f(x) = \begin{cases} 3x - x^3 & \text{if } 0 \leq x \leq 2, \\ -2x & \text{if } -1/2 \leq x < 0. \end{cases}$

Find the absolute maximum and the absolute minimum of $f(x)$ over the interval $[-1/2, 2]$.

$$\frac{d}{dx}(3x - x^3) = 3 - 3x^2 = 0 \text{ when } x = -1 \text{ or } x = 1.$$ However, only $x = 1$ is in $[0, 2]$. The critical points are 1 and 0, where the function has a corner. When we look at endpoints and critical points, we get

$$f(-1/2) = 1 \quad f(0) = 0 \quad f(1) = 2 \quad f(2) = -2.$$ The absolute maximum is at $x = 1$. The absolute minimum is at $x = 2$.

(2) Assume $f(x) = \begin{cases} x^3 - 3x & \text{if } 0 \leq x \leq 3/2, \\ x & \text{if } -1 \leq x < 0. \end{cases}$

Find the absolute maximum and the absolute minimum of $f(x)$ over the interval $[-1, 3/2]$.

$$\frac{d}{dx}(x^3 - 3x) = 3x^2 - 3 = 0 \text{ when } x = -1 \text{ or } x = 1.$$ However, only $x = 1$ is in $[0, 3/2]$. The critical points are 1 and 0, where the function has a corner. When we look at endpoints and critical points, we get

$$f(-1) = -1 \quad f(0) = 0 \quad f(1) = -2 \quad f(3/2) = \frac{27}{8} - \frac{9}{2} = -\frac{9}{8}.$$ The absolute maximum is at $x = 0$. The absolute minimum is at $x = 1$.

In problems (3)-(10), find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing, find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down, find the relative maxima and the relative minima of $f(x)$, find the inflection points of $f(x)$, and find the horizontal and vertical asymptotes of $f(x)$.

(3) $f(x) = \frac{x^5}{5} - \frac{4x^3}{3} + 3x$

We get $f'(x) = x^4 - 4x^2 + 3 = (x^2 - 1)(x^2 - 3)$ and $f''(x) = 4x^3 - 8x = 4x(x^2 - 2)$.

The function is increasing on $(-\infty, -\sqrt{3}), (-1, 1), (\sqrt{3}, \infty)$.

The function is decreasing on $(-\sqrt{3}, -1), (1, \sqrt{3})$.

There are relative maxima at $x = -\sqrt{3}$ and $x = 1$. There are relative minima at $x = -1$ and $x = \sqrt{3}$.

The function is concave up on $(-\sqrt{2}, 0), (\sqrt{2}, \infty)$.

The function is concave down on $(-\infty, -\sqrt{2}), (0, \sqrt{2})$.

There are inflection points at $x = -\sqrt{2}, x = 0, x = \sqrt{2}$.
(4) $f(x) = xe^{-x^2}$

We get

\[ f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1 - 2x^2)e^{-x^2} \]
\[ f''(x) = -4xe^{-x^2} + (1 - 2x^2)(-2x)e^{-x^2} = 2x(2x^2 - 3)e^{-x^2} \]

The function is increasing on $(-1/\sqrt{2}, 1/\sqrt{2})$.
The function is decreasing on $(-\infty, -1/\sqrt{2})$, $(1/\sqrt{2}, \infty)$.
There is a relative maximum at $x = 1/\sqrt{2}$.
There is a relative minimum at $x = -1/\sqrt{2}$.
The function is concave up on $(-\sqrt{3/2}, 0)$, $(\sqrt{3/2}, \infty)$.
The function is concave down on $(-\infty, -\sqrt{3/2})$, $(0, \sqrt{3/2})$.
There are inflection points at $x = -\sqrt{3/2}$, $x = 0$, $x = \sqrt{3/2}$.
$y = 0$ is a horizontal asymptote.

(5) $f(x) = x^3\ln x$

We have $f'(x) = 3x^2\ln x + x^2 = x^2(3\ln x + 1)$, $f''(x) = 2x(3\ln x + 1) + x^2(3/x) = 6\ln x + 5x = x(6\ln x + 5)$.
The function is increasing on $(e^{-1/3}, \infty)$ and decreasing on $(0, e^{-1/3})$.
There is a relative minimum at $x = e^{-1/3}$.
The function is concave up on $(e^{-5/6}, \infty)$ and concave down on $(0, e^{-5/6})$.
There is an inflection point at $x = e^{-5/6}$.

(6) $f(x) = x + \sin(2x)$

We have $f'(x) = 1 + 2\cos(2x)$ and $f''(x) = -4\sin(2x)$.
The function is increasing on $(-\pi/3 + n\pi, \pi/3 + n\pi)$, where $n$ is an integer.
The function is decreasing on $(\pi/3 + n\pi, 2\pi/3 + n\pi)$, where $n$ is an integer.
There are relative maxima at $x = \pi/3 + n\pi$, where $n$ is an integer.
There are relative minima at $x = -\pi/3 + n\pi$, where $n$ is an integer.
The function is concave up on $(\pi/2 + n\pi, \pi + n\pi)$, where $n$ is an integer.
The function is concave down on $(n\pi, \pi/2 + n\pi)$, where $n$ is an integer.
There are an inflection points at $x = n\pi/2$, where $n$ is an integer.

(7) $f(x) = \frac{1}{4 + x^2}$. We have

\[ f'(x) = -(4 + x^2)^{-2}2x = \frac{-2x}{(4 + x^2)^2} \]
\[ f''(x) = \frac{-2(4 + x^2) - (2x)(2)(4 + x^2)2x}{(4 + x^2)^4} = \frac{-2(4 + x^2) + 8x^2}{(4 + x^2)^3} = \frac{6x^2 - 8}{(4 + x^2)^3} \]

The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. There is a maximum at $x = 0$, $y = 0$ is a horizontal asymptote.
The function is concave up on $(-\sqrt{4/3}, 0)$ and $(\sqrt{4/3}, \infty)$.
The function is concave down on $(-\sqrt{4/3}, \sqrt{4/3})$.
There are inflection points at $x = -\sqrt{4/3}$ and $x = \sqrt{4/3}$. 

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(8) \( f(x) = \frac{1}{4 - x^2} \). We have

\[
\begin{align*}
    f'(x) &= (2x)(4 - x^2)^{-2} = \frac{2x}{(4 - x^2)^2} \\
    f''(x) &= \frac{2(4 - x^2)^2 - (2x)2(4 - x^2)(-2x)}{(4 - x^2)^4} = \frac{2(4 - x^2) + 8x^2}{(4 - x^2)^3} = \frac{8 + 6x^2}{(4 - x^2)^3}
\end{align*}
\]

The function is increasing on \((0, 2), (2, \infty)\).
The function is decreasing on \((\infty, -2), (-2, 0)\).
There is a relative minimum at \(x = 0\). \(y = 0\) is a horizontal asymptote. \(x = \pm 2\) are vertical asymptotes.
The function is concave up on \((-2, 2)\).
The function is concave down on \((-\infty, -2), (2, \infty)\).

(9) \( f(x) = \frac{x}{4 + x^2} \). We have

\[
\begin{align*}
    f'(x) &= \frac{(4 + x^2) - x(2x)}{(4 + x^2)^2} = \frac{4 - x^2}{(4 + x^2)^2} \\
    f''(x) &= \frac{-2x(4 + x^2)^2 - (4 - x^2)2(4 + x^2)(2x)}{(4 + x^2)^4} = \frac{-2x(4 + x^2) - (4 - x^2)4x}{(4 + x^2)^3} \\
    &= \frac{2x(x^2 - 12)}{(4 + x^2)^3}
\end{align*}
\]

The function is increasing on \((-2, 2)\). The function is decreasing on \((\infty, -2), (2, \infty)\).
There is a relative maximum at \(x = 2\). There is a relative minimum at \(x = -2\).
\(y = 0\) is a horizontal asymptote.
The function is concave up on \((-\sqrt{12}, 0), (\sqrt{12}, \infty)\), \((-\sqrt{4/3}, \sqrt{4/3})\).
There are inflection points at \(x = -\sqrt{4/3}\) and \(x = \sqrt{4/3}\).
(8) \( f(x) = \frac{1}{4-x^2} \). We have

\[
\begin{align*}
  f'(x) &= (2x)(4-x^2)^{-2} = \frac{2x}{(4-x^2)^2} \\
  f''(x) &= \frac{2(4-x^2)^2 - (2x)2(4-x^2)(-2x)}{(4-x^2)^4} = \frac{2(4-x^2) + 8x^2}{(4-x^2)^3} = \frac{8 + 6x^2}{(4-x^2)^3}
\end{align*}
\]

The function is increasing on \((0,2), (2,\infty)\).
The function is decreasing on \((-\infty,-2), (-2,0)\).
There is a relative minimum at \(x = 0\). \(y = 0\) is a horizontal asymptote. \(x = \pm 2\) are vertical asymptotes.
The function is concave up on \((-2,2)\).
The function is concave down on \((-\infty,-2), (2,\infty)\).

(9) \( f(x) = \frac{x}{4+x^2} \). We have

\[
\begin{align*}
  f'(x) &= \frac{(4+x^2) - x(2x)}{(4+x^2)^2} = \frac{4-x^2}{(4+x^2)^2} \\
  f''(x) &= \frac{-2x(4+x^2)^2 - (4-x^2)2(4+x^2)(2x)}{(4+x^2)^4} = \frac{-2x(4+x^2) - (4-x^2)4x}{(4+x^2)^3} \\
  &= \frac{2x(x^2 - 12)}{(4+x^2)^3}
\end{align*}
\]

The function is increasing on \((-2,2)\). The function is decreasing on \((-\infty,-2), (2,\infty)\).
There is a relative maximum at \(x = 2\). There is a relative minimum at \(x = -2\).
\(y = 0\) is a horizontal asymptote.
The function is concave up on \((-\sqrt{12},0), (\sqrt{12},\infty)\). The function is concave down on \((-\infty, -\sqrt{12}), (0, \sqrt{12})\). There are inflection points at \(x = -\sqrt{12}, x = 0, x = \sqrt{12}\).

(10) \( f(x) = \frac{x}{4-x^2} \). We have

\[
\begin{align*}
  f'(x) &= \frac{(4-x^2) - x(-2x)}{(4-x^2)^2} = \frac{4+x^2}{(4-x^2)^2} \\
  f''(x) &= \frac{2x(4-x^2)^2 - (4+x^2)2(4-x^2)(-2x)}{(4-x^2)^4} = \frac{2x(4-x^2) + 4x(4+x^2)}{(4-x^2)^3} \\
  &= \frac{2x(12+x^2)}{(4-x^2)^3}
\end{align*}
\]

The function is increasing on \((-\infty,-2), (-2,2), (2,\infty)\).
\(y = 0\) is a horizontal asymptote. \(x = \pm 2\) are vertical asymptotes.
The function is concave up on \((-\infty,-2), (0,2)\). The function is concave down on \((-2,0), (2,\infty)\). There is an inflection point at \(x = 0\).
(11) Evaluate the derivatives of the following functions:

\[
\begin{align*}
\sin^{-1}(x^3 + 1) & \quad \tan^{-1}(e^{2x} + x) & \quad \sec^{-1}(x^5 + x^3) & \quad \ln(1 + x^4 + x^2)
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} \sin^{-1}(x^3 + 1) &= \frac{3x^2}{\sqrt{1 - (x^3 + 1)^2}} \\
\frac{d}{dx} \tan^{-1}(e^{2x} + x) &= \frac{2e^{2x} + 1}{1 + (e^{2x} + x)^2} \\
\frac{d}{dx} \sec^{-1}(x^5 + x^3) &= \frac{5x^4 + 3x^2}{|x^5 + x^3|\sqrt{(x^5 + x^3)^2 - 1}} \\
\frac{d}{dx} \ln(1 + x^4 + x^2) &= \frac{4x^3 + 2x}{1 + x^4 + x^2}
\end{align*}
\]

(12) Find an equation for the line that is tangent to the curve \(x^2y^4 + x^4y^2 = 20\) at the point \((1, 2)\) on that curve.

We have \(\frac{d}{dx}(x^2y^4 + x^4y^2) = \frac{d}{dx}20 = 0\), hence

\[2xy^4 + x^2(4y^3y') + 4x^3y^2 + x^4(2yy') = 0\]

where \(y' = \frac{d}{dx}y\). Solving for \(y'\), we get

\[(4x^2y^3 + 2x^4y)y' + (2xy^4 + 4x^3y^2) = 0\]

hence

\[y' = -\frac{2xy^4 + 4x^3y^2}{4x^2y^3 + 2x^4y} = \frac{2 \cdot 1 \cdot 2^4 + 4 \cdot 1^3 \cdot 2^2}{4 \cdot 1^2 \cdot 2^3 + 2 \cdot 1^4 \cdot 2} = \frac{48}{36} = \frac{4}{3}\]

The equation of the tangent line is \(y - 2 = \frac{4}{3}(x - 1)\).

(13) Evaluate \(\frac{d}{dx}((\sin x)^{\cos x})\).

We have \(\frac{d}{dx}(\ln ((\sin x)^{\cos x})) = \frac{d}{dx}(\cos x \ln(\sin x)) = -(\sin x)\ln(\sin x) + (\cos x) \frac{\cos x}{\sin x}\)

Therefore, \(\frac{d}{dx}((\sin x)^{\cos x}) = (\sin x)^{\cos x} \left[-(\sin x)\ln(\sin x) + (\cos x) \frac{\cos x}{\sin x}\right]\)
Evaluate the limits in problems (15)-(23). You may use L'Hôpital's Rule.

(14) \( \lim_{x \to \infty} \frac{x^3}{e^{x/2}} \)

We have \( \lim_{x \to \infty} \frac{x^3}{e^{x/2}} = \lim_{x \to \infty} \frac{3x^2}{(1/2)e^{x/2}} = \lim_{x \to \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \to \infty} \frac{6}{(1/8)e^{x/2}} = 0 \)

(15) \( \lim_{x \to 4} \left( \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right) \)

We have

\[
\lim_{x \to 4} \left( \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right) = \lim_{x \to 4} \left( \frac{\sqrt{x} + 2}{(\sqrt{x} - 2)(\sqrt{x} + 2) - 4} \right) = \lim_{x \to 4} \left( \frac{\sqrt{x} + 2}{x - 4} - \frac{4}{x - 4} \right) = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(1/2)x^{-1/2}}{1} = \frac{1}{4}
\]

(16) \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \)

We have \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0 \)

(17) \( \lim_{x \to 0} (1 + 3x)^{1/x} \)

We have \( \lim_{x \to 0} \left( \ln \left( (1 + 3x)^{1/x} \right) \right) = \lim_{x \to 0} \frac{\ln(1 + 3x)}{x} = \lim_{x \to 0} \frac{3}{1 + 3x} = 3 \)

This implies \( \lim_{x \to 0} (1 + 3x)^{1/x} = e^3 \)

(18) \( \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x \)

We have

\[
\lim_{x \to \infty} \left( \ln \left( 1 + \frac{2}{x} \right)^x \right) = \lim_{x \to \infty} x \ln \left( 1 + \frac{2}{x} \right) = \lim_{x \to \infty} \frac{\ln \left( 1 + \frac{2}{x} \right)}{1/x} = \lim_{x \to \infty} \frac{-2/x^2}{1+2/x} = \lim_{x \to \infty} \frac{2}{1+2/x} = 2
\]

Therefore, \( \lim_{x \to \infty} \left( 1 + \frac{2}{x} \right)^x = e^2 \).

\[\lim_{x \to \pi/2} (\tan x - \sec x)\]

We have \(\lim_{x \to \pi/2} (\tan x - \sec x) = \lim_{x \to \pi/2} \left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right)\)

\[= \lim_{x \to \pi/2} \frac{\sin x - 1}{\cos x} = \lim_{x \to \pi/2} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0\]

\(\lim_{x \to 0} \frac{1 - \cos(5x)}{1 - \cos(3x)}\)

We have \(\lim_{x \to 0} \frac{1 - \cos(5x)}{1 - \cos(3x)} = \lim_{x \to 0} \frac{5 \sin(5x)}{3 \sin(3x)} = \lim_{x \to 0} \frac{25 \cos(5x)}{9 \cos(3x)} = \frac{25}{9}\)

\(\lim_{x \to \infty} \left(\ln(5x^2 - x + 6) - \ln(3x^2 + 7x - 1)\right)\)

We have \(\lim_{x \to \infty} \left(\ln(5x^2 - x + 6) - \ln(3x^2 + 7x - 1)\right) = \lim_{x \to \infty} \ln \left(\frac{5x^2 - x + 6}{3x^2 + 7x - 1}\right) = \ln \left(\frac{5}{3}\right)\)

(23) A spherical balloon is being inflated at a rate of 7 cubic inches per second. How fast is the radius of the balloon increasing when the radius is 5 inches?

Let \(t\) be time, let \(R\) be the radius of the balloon, and let \(V\) be the volume of the balloon.

Differentiating the equation \(V = \frac{4}{3} \pi R^3\), we get \(\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}\). Substituting the given information into this, we get \(7 = 4\pi R^2 \frac{dR}{dt}\), Which gives \(\frac{dR}{dt} = \frac{7}{4\pi R^2}\). The answer to the question is that the radius is increasing at a rate of \(\frac{7}{100\pi}\) inches per second.

(24) A spherical balloon is being inflated at a rate of 7 cubic inches per second. What is the radius of the balloon when the surface area of the balloon is increasing at a rate of 5 square inches per second?

Let \(t\) be time, let \(R\) be the radius of the balloon, let \(V\) be the volume of the balloon, and let \(S\) be the surface area of the balloon. Differentiating the equation \(V = \frac{4}{3} \pi R^3\), we get \(\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}\). Differentiating the equation \(S = 4\pi R^2\), we get \(\frac{dS}{dt} = 8\pi R \frac{dR}{dt}\).

Substituting the given information into this, we get \(7 = 4\pi R^2 \frac{dR}{dt}\) and \(5 = 8\pi R \frac{dR}{dt}\). If we divide \(7 = 4\pi R^2 \frac{dR}{dt}\) by 5 = \(8\pi R \frac{dR}{dt}\) then we get \(\frac{dR}{dt} = \frac{\frac{7}{5}}{\frac{8\pi R}{5}} = \frac{R}{2}\) hence \(R = \frac{14}{5}\). The answer to the question is radius = \(\frac{14}{5}\) inches.
(25) A North-South road meets an East-West road at an intersection. At a certain moment, a car on the North-South road is 4 miles north of the intersection and is traveling north at 55 miles per hour. At the same moment, a truck on the East-West road is 3 miles east of the intersection and is traveling east at 45 miles per hour. How fast is the distance between the car and the truck increasing at that moment?

Let \( t \) be time. Let \( x \) be the distance from the intersection to the location of the truck. Let \( y \) be the distance from the intersection to the location of the car. Let \( z \) be the distance between the truck and the car. Differentiating the equation \( z^2 = x^2 + y^2 \), we get \( 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \), which simplifies to \( z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \). Substituting the given information into this, we get \( 5 \frac{dz}{dt} = 3 \cdot 45 + 4 \cdot 55 \), where we got the 5 from 3, 4 and the Pythagorean Theorem. Solving, we get \( \frac{dz}{dt} = \frac{3 \cdot 45 + 4 \cdot 55}{5} = 71 \). The distance between the car and the truck is increasing at the rate of 71 miles per hour.

(26) A water tank is a cone with a height of 20 feet and a radius of 5 feet at its base. A spigot at the bottom of the tank is open, and the water is flowing out of the tank at a rate of 7 cubic feet per second. How fast is the water level in the tank dropping when the water is 10 feet high?

Let \( t \) be time. Let \( H \) be the height of the water in the tank. Let \( V \) be the volume of the water in the tank. We need to find a formula for \( V \) in terms of \( H \). In order to do this, we will first find out the volume of the cone that consists of the portion of the tank that has no water in it. The portion of the tank that is just air is really a cone because we are assuming that the tank is sitting with its base on the floor, which is much more stable than having the tank balancing on its thin vertex point. Suppose that this cone has height \( h \) and radius \( r \) at its base. Similar triangles tell us that we must have \( \frac{r}{h} = \frac{5}{20} \), which is \( r = h/4 \). We know \( h = 20 - H \). All this implies \( r = (20 - H)/4 \). Now we know that the cone without water has volume equal to \( (1/3)\pi r^2 h = (1/3)\pi ((20 - H)/4)^2 (20 - H) = (1/48)\pi (20 - H)^3 \). Since \( V \) equals the volume of the tank minus the volume of the cone with no water, we obtain

\[
V = \frac{1}{3} \pi 5^2 20 - (1/48)\pi (20 - H)^3 = \frac{500}{3} \pi - (1/48)\pi (20 - H)^3
\]

Differentiating this equation, we get

\[
\frac{dV}{dt} = -\frac{1}{16} \pi (20 - H)^2 \left( -\frac{dH}{dt} \right) = \frac{1}{16} \pi (20 - H)^2 \frac{dH}{dt}
\]

Substituting given information into this last equation, we get

\[
-7 = \frac{1}{16} \pi (20 - 10)^2 \frac{dH}{dt} = \frac{100}{16} \pi \frac{dH}{dt}
\]

which gives \( \frac{dH}{dt} = -\frac{112}{100\pi} \). The water level is decreasing at the rate of \( \frac{112}{100\pi} \) feet per second.
Approximate $\sqrt{9.001}$ using linearization at 9.

The formula for linearization is $L(x) = f(a) + f'(a)(x - a)$. When $f(x) = \sqrt{x}$ and $a = 9$ we get $L(x) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x - 9) = 3 + \frac{1}{6}(x - 9)$. Then we get $L(9.001) = 3 + \frac{1}{6}(0.001) = 3 + \frac{1}{6000}$.

Find the linearization $L(x)$ of the function $f(x) = \sin x$ centered at $\pi/3$.

The linearization is $L(x) = f(a) + f'(a)(x - a)$ with $f(x) = \sin x$ and $a = \pi/3$. This gives

$$L(x) = \sin(\pi/3) + \cos(\pi/3)(x - \pi/3) = \sqrt{3}/2 + (1/2)(x - \pi/3)$$

The radius of a pizza plate is known with an accuracy of 0.1%. With what accuracy do we know the area of the pizza plate?

If $R$ is the radius of the pizza plate and $A$ is its area, then $A = \pi R^2$. Let $\Delta R$ be the uncertainly in $R$ and let $\Delta A$ be the uncertainly in $A$. From

$$\frac{\Delta A}{\Delta R} \approx \frac{dA}{dR} = 2\pi R$$

we get $\Delta A \approx 2\pi R \Delta R$.

Dividing this by $A = \pi R^2$ we get

$$\frac{\Delta A}{A} \approx \frac{2\pi R \Delta R}{\pi R^2}$$

which is $\frac{\Delta A}{A} \approx 2 \frac{\Delta R}{R}$.

If the radius of the pizza plate is known with an accuracy of 0.1% , then $\frac{\Delta R}{R} = 0.001$. Substituting this into $\frac{\Delta A}{A} \approx 2 \frac{\Delta R}{R}$, we get $\frac{\Delta A}{A} \approx 2(0.001) = 0.002$. This says that the area of the pizza plate is known with an accuracy of 0.2%.

Assume $f(x) = \sqrt{x}$. Find a number $c$ such that $5 < c < 7$ and $f'(c) = \frac{f(7) - f(5)}{7 - 5}$.

We need to find a number $c$ such that $5 < c < 7$ and $\frac{1}{2\sqrt{c}} = \frac{\sqrt{7} - \sqrt{5}}{2}$. The only solution is $c = \frac{1}{12 - 2\sqrt{35}} = 3 + \frac{\sqrt{35}}{2}$. 

9
(31) Assume \( f(x) = x^3 - 3x \). Find all numbers \( c \) such that \(-2 < c < 3\) and \( f'(c) = \frac{f(3) - f(-2)}{3 - (-2)} \).

We need to find numbers \( c \) such that \(-2 < c < 3\) and

\[
3c^2 - 3 = \frac{(3^3 - 3(3)) - ((-2)^3 - 3(-2))}{3 - (-2)},
\]

which is \(3(c^2 - 1) = 4\). The solutions are \( c = \pm \sqrt{7}/3\).

(32) Find two positive numbers \( x \) and \( y \) such that \( x + y = 10 \) and \( x^2y \) is as large as possible.

Since \( y = 10 - x \), we are maximizing \( x^2(10 - x) = 10x^2 - x^3 \) over \([0, 10]\). The critical point of \( 10x^2 - x^3 \) in \((0, 10)\) is \( \frac{20}{3} \). We see that \( 10x^2 - x^3 \) equals 0 at the endpoints \( x = 0, x = 10 \), but \( 10x^2 - x^3 \) is positive at the critical point \( x = \frac{20}{3} \). This implies that the maximum occurs at \( x = \frac{20}{3} \).

(33) What is the largest possible volume of a cone with the property that 10 inches equals the distance from the vertex to any point on the circumference of the base?

Let \( R \) be the radius of the base of the cone, and let \( H \) be the height of the cone. We are maximizing the volume \( \frac{1}{3} \pi R^2 H \) subject to the condition \( R^2 + H^2 = 10^2 \), which is \( R^2 = 100 - H^2 \). This means that we are maximizing \( \frac{1}{3} \pi (100 - H^2)H \) on the interval \( 0 \leq H \leq 10 \). The critical point of \( \frac{1}{3} \pi (100 - H^2)H \) in the interval \((0, 10)\) is \( H = \frac{10}{\sqrt{3}} \). We see that \( \frac{1}{3} \pi (100 - H^2)H \) equals 0 at the endpoints \( H = 0, H = 10 \), but \( \frac{1}{3} \pi (100 - H^2)H \) is positive at the critical point \( H = \frac{10}{\sqrt{3}} \). This implies that the maximum occurs at \( H = \frac{10}{\sqrt{3}} \).
(34) What is the largest possible volume of a cylindrical can with surface area equal to 100 square inches?

Let $R$ be the radius of the can, and let $H$ be the height of the can. The volume of the can is $\pi R^2 H$. The surface area of the can is $2\pi R^2 + 2\pi RH$. We are maximizing $\pi R^2 H$ subject to the condition $2\pi R^2 + 2\pi RH = 100$. We see that $2\pi R^2 = 100 - 2\pi RH \leq 100$ implies $R \leq \frac{10}{\sqrt{2\pi}}$. We also see that $2\pi R^2 + 2\pi RH = 100$ is equivalent to $H = \frac{100 - 2\pi R^2}{2\pi R} = \frac{100}{2\pi R} - R$. This means that we are maximizing $\pi R^2 \left(\frac{100}{2\pi R} - R\right)$ over the interval $0 \leq R \leq \frac{10}{\sqrt{2\pi}}$. The critical point of $\pi R^2 \left(\frac{100}{2\pi R} - R\right)$ is $R = \frac{10}{\sqrt{6\pi}}$. We see that $\pi R^2 \left(\frac{100}{2\pi R} - R\right)$ equals $0$ at the endpoints $R = 0$, $R = \frac{10}{\sqrt{2\pi}}$, but $\pi R^2 \left(\frac{100}{2\pi R} - R\right)$ is positive at the critical point $R = \frac{10}{\sqrt{6\pi}}$. This implies that the maximum occurs at the critical point $R = \frac{10}{\sqrt{6\pi}}$.

(35) What is the smallest possible surface area of a cylindrical can with volume equal to 1000 cubic inches?

Let $R$ be the radius of the can, and let $H$ be the height of the can. The volume of the can is $\pi R^2 H$. The surface area of the can is $2\pi R^2 + 2\pi RH$. We are minimizing $2\pi R^2 + 2\pi RH$ subject to the condition $\pi R^2 H = 1000$. Since $\pi R^2 H = 1000$ is equivalent to $H = \frac{1000}{\pi R^2}$, we are minimizing $2\pi R^2 + 2\pi R \left(\frac{1000}{\pi R^2}\right)$ over the open interval $0 < R < \infty$. The critical point of $2\pi R^2 + 2\pi R \left(\frac{1000}{\pi R^2}\right)$ is $R = \frac{10}{(2\pi)^{1/3}}$. This critical point gives us an absolute minimum of $2\pi R^2 + 2\pi R \left(\frac{1000}{\pi R^2}\right)$ over $0 < R < \infty$ because the second derivative of $2\pi R^2 + 2\pi R \left(\frac{1000}{\pi R^2}\right)$ is always positive, which means that $2\pi R^2 + 2\pi R \left(\frac{1000}{\pi R^2}\right)$ is concave up for all $R > 0$. 

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