Math 251 Multivariable Calculus

PRACTICE PROBLEMS—EXAM 1

Please note that this is not a practice exam; in particular, there are more problems here than will be on the exam. Moreover, although these problems are generally similar to exam problems, it is possible that the exam will contain some problems quite different from any here.

The answers are not guaranteed.

Ch. 12
1. Write the vector \( \mathbf{u} = \langle 2, -3, 1 \rangle \) in the form \( \mathbf{v} + \mathbf{w} \), where \( \mathbf{v} \) is a vector parallel to the vector \( \langle 1, 1, -1 \rangle \) and \( \mathbf{w} \) is a vector perpendicular to \( \mathbf{v} \).

2. Find parametric equations of the line through the point \( (1, -3, 1) \) and parallel to the line of intersection of the planes \( 2x + 2y - z = 1 \) and \( 3x - y = 0 \).

3. (a) Find a parametrization of the line passing through the points \( (1, 2, 3) \) and \( (3, 5, 7) \).
   (b) Let \( g(t) \) be the square of the distance from the origin to a point on the line at time \( t \). Find a formula for \( g(t) \), and use it to find the point on the line that is closest to the origin.

4. (a) Show that the line \( x = 2 - t, y = 1 + 2t, z = 3 + t \) intersects the line \( x = 3 + 2s, y = -3s, z = 5 + s \), and find the point of intersection.
   (b) Find the equation of the plane containing both of these lines.

5. Find vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) such that \( \mathbf{u} = \langle 1, -2, -1 \rangle, \mathbf{v} = \langle 0, a, b \rangle \) for some \( a, b \), and each pair of vectors (\( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{u} \) and \( \mathbf{w} \), and \( \mathbf{v} \) and \( \mathbf{w} \)) are orthogonal.

6. (a) Find the equation of the plane through the points \( Q = (1, 3, 2), R = (2, 1, 4), \) and \( S = (1, 5, 6) \).
   (b) Use the dot product to find the cosine of the angle \( \theta \) at \( Q \) of the triangle \( QRS \).
   (c) Use the cross product to find the sine of \( \theta \), and verify that \( \cos^2 \theta + \sin^2 \theta = 1 \).

Ch. 13
7. (a) Parameterize the circle \( (x + 3)^2 + y^2 = 4 \).
   (b) Parameterize the curve which is the intersection of the plane \( 2x + 4y + z = 4 \) with the surface \( z = x^2 + y^2 \). Hint: eliminate \( z \).

8. The acceleration of a particle in the plane is \( \mathbf{a} = 2\mathbf{i} + 8\mathbf{j} \).
   (a) Find the particle’s position as a function of \( t \) if its velocity at time \( t = 0 \) is \( \mathbf{i} - \mathbf{j} \) and its position at time \( t = 0 \) is \( 2\mathbf{i} + 3\mathbf{j} \).
   (b) Find the speed of the particle at time \( t = 1 \).

9. Find the length of the curve \( \mathbf{r}(t) = \langle (t^2 - 1), (3t^2 - 2), (t^2 + 1) \rangle \) from \( t = 1 \) to \( t = 2 \).
   Also find the unit tangent vector to the curve at \( t = 1 \).

Ch. 14
10. Suppose \( f(x, y) = (5x^2 - 2y^2)/(7x^2 + 3y^2) \) if \( (x, y) \neq (0, 0) \). Find the limit of \( f(x, y) \) as \( (x, y) \) approaches \( (0, 0) \) along (i) the \( x \)-axis, (ii) the \( y \)-axis. Does \( \lim_{(x,y)\to(0,0)} f(x,y) \) exist? Explain.

11. Sketch any 3 level curves of the function \( f(x, y) = 2 - x^2 - y \). Label each curve with the appropriate function value.
12. Let \( f(x, y) = \sqrt{(x-a)^2 + (y-b)^2} \). Show that for any \((x, y) \neq (a, b)\), we have
\[
\sqrt{\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} = 1.
\]

13. Find all first and second order partial derivatives of \( f(x, y) = x^2(y+1) + y^2 e^{2x} \).
\( f_x = 2x(y+1) + 2y^2 e^{2x}; \quad f_y = x^2 + 2ye^{2x}; \quad f_{xx} = 2(y+1) + 4y^2 e^{2x}; \quad f_{xy} = 2x + 4ye^{2x}; \quad f_{yy} = 2e^{2x} \)

14. The elevation on a particular hill at the point \((x, y)\) is given by the function \( h(x, y) = c - (x-3)^2 - (y-4)^2 \).
(a) Write the linearization to \( h \) at \((2, 3, 2)\).
(b) If the tangent plane to the graph of \( h \) at \((2, 3, 2)\) passes through the origin, what is the height of the hill?
\[
L(x, y) = c-1+1.2(x-2.4)+1.6(y-3.2): 8
\]

15. (a) Define precisely what it means for a function \( f(x, y) \) to be **differentiable** at a point \((x_0, y_0)\).
(b) Demonstrate, using the definition, that \( f(x, y) = x^2 - 6x + y^2 + 4y \) is differentiable at \((3, -2)\). [Hint: first show that the linearization, \( L(x, y) \), at this point is a constant.]

16. Suppose \( f(x, y) = xe^y \). Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((2, 0, 2)\). Use the linearization to find an approximation to \( f(2.1, 0.2) \).
\[
z - 2 = (x - 2) + 2y; 2.5
\]

17. Find the derivative of \( f(x, y, z) = \cos(xy) + e^{yz} + \ln(zx) \) at the point \( P_0 = (1, 0, 1/2) \) in the direction of the vector \( \mathbf{A} = i + j - \sqrt{2}k \). In what direction does \( f \) decrease most rapidly at \( P_0 \)? Find equations for the tangent plane and normal line to the surface \( f(x, y, z) = 2 - \ln 2 \) at \( P_0 \).
\[
3/4 - \sqrt{2}; \text{ direction of } -((1+(1/2))j+2k); 2x+y+4z=4; x=1+t, y=t/2, z=(1/2)+2t
\]

18. Let \( f(x, y, z) \) be a differentiable function and suppose that \( c(t) \) is a path which lies on the surface \( f(x, y, z) = 17 \). If \( c(5) = \langle 1, 4, 2 \rangle \) show that \( c'(5) \) is orthogonal to \( \nabla f|_{(1,4,2)} \).

19. For \( f(x, y) = A - (x^2 + Bx + y^2 + Cy) \), what values of \( A, B, \) and \( C \) give \( f \) a local maximum of 15 at the point \((-2, 1)\)?
\[
6, 4, -2
\]

20. At the point \((1, 3)\), suppose \( f_x = f_y = 0 \) and \( f_{xx} < 0, f_{yy} < 0, f_{xy} = 0 \). Draw a possible contour diagram.

21. Let \( S \) denote the surface \( x^2 + y^2 - 3z^2 = 10 \), and \( P \) the point \((2, -3, 1)\), which is on \( S \). Use implicit differentiation to write an equation for the tangent plane to \( S \) at \( P \).
\[
z = 1 + \frac{3}{2} (x-2) - (y+3)
\]

22. Let \( f(x, y, z) = x^2 + y^2 + z^2 \). At the point \((1, 2, 1)\), find the rate of change of \( f \) in the direction perpendicular to the plane \( 4x - 4y + 7z = 3 \), pointing away from the origin.
\[
2/3
\]

23. Let \( f(x, y, z) = x^2 - y/z^2 \), and let \( P = (1/\sqrt{2}, 2\sqrt{2}, \sqrt{2}) \).
(a) Calculate \( \nabla f_P \).
(b) Evaluate the directional derivative at \( P \) in the direction of \( \nabla f_P \).
(c) Evaluate the directional derivative at \( P \) in the direction of a vector making an angle of \( 2\pi/3 \) with \( \nabla f_P \).
\[
(\sqrt{2}, -1/2, 2); 5/2; -5/4
\]

24. A function \( f(x, y) \) has gradient \( \langle -3, 4 \rangle \) at the point \( P = (-1, 3) \). Write an equation for the tangent line at \( P \) to the level curve of \( f \) passing through \( P \).
\[
y - 3 = \frac{1}{4} (x+1)
\]