Mathematics 552 **Algebra II** **Spring 2017**

The set of k-dimensional subspaces in an m-dimensional vector space

Let K be a field and M a vector space over K of dimension m. For an integer k with $0 \leq k \leq m$ let $G(k, M)$ be the set of k-dimensional subspaces of M. We also write $G(k, m)$ for the set $G(k, K^m)$ and refer to it as the Grassmannian of k-planes in M in honor of Grassmann who introduced exterior algebras.

Theorem. *There is a one-to-one map of the set $G(k, M)$ to the set of lines in $\Lambda^k(M)$ spanned by decomposable vectors of the form $v_1 \wedge \cdots \wedge v_k$.*

Proof.

Suppose that V is a k-dimensional subspace of M with basis $v_1, \ldots, v_k$. Then $v_1 \wedge \cdots \wedge v_k \in \Lambda^k(M)$ is decomposable. If $v'_1, \ldots, v'_k$ is another basis of V then the isomorphism ϕ of V with itself obtained by sending v_i to v'_i induces the map multiplication by $\det(\phi)$ on $\Lambda^k(V)$ so that $v_1 \wedge \cdots \wedge v_k = \det(\phi) v'_1 \wedge \cdots \wedge v'_k$. Thus changing the basis does not alter the line associated to V so the map is well defined.

Given $\omega \in \Lambda^k(M)$ consider the linear map $\phi_\omega(v) = v \wedge \omega$ from V to $\Lambda^{k+1}V$. Let $w_1, \ldots, w_r$ be a basis for the kernel of this map and extend this linearly independent set to a basis $w_1, \ldots, w_n$ of M. Write $\omega = \sum_I a_I w_I$ where I is a set with k elements $i_1 < i_2 \cdots < i_k$ from the integers $1, \ldots, m$ and $w_I = w_{i_1} \wedge \cdots \wedge w_{i_k}$. Since w_1 is in the kernel of ϕ_ω we have $w_1 \wedge \omega = \sum_I a_I w_1 \wedge w_I = 0$. Thus $a_I = 0$ if $1 \notin I$, so that each w_I appearing with nonzero coefficient is of the form $w_1 \wedge w_{I-1}$ so that $\omega = w_1 \wedge \omega_1$ for some $\omega_1 \in \Lambda^{k-1}(M)$. Similarly ω is divisible by $w_1 \wedge \cdots \wedge w_r$ so that $r \leq k$ and if $r = k$ then ω is decomposable. Thus given any decomposable element $\omega = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(M)$ the kernel of ϕ_ω is the k-dimensional subspace of M spanned by v_i, so a decomposable ω determines a unique subspace V, showing that the map is one-to-one.

The coordinates of ω with respect to the basis v_I are called the Plucker coordinates of the subspace V. So the lines spanned by decomposable elements coordinatize the set $G(k, M)$. If A is the $k \times m$ matrix expressing a basis of $V \subset M$ in terms of a chosen basis of M then the coefficient A_I of v_I in this expression is the determinant of the submatrix formed by columns indexed by elements of I.

Proposition. *There exists a collection of polynomials in the variables x_I indexed by all I an ordered list of k elements chosen from $[1, m]$ such that an element $\omega = \sum_I a_I v_I \in \Lambda^k(V)$ is decomposable if and only if the coordinates a_I are zeros of each polynomial in the collection.*

Proof.

Let v_i be a basis of M. An element $\omega = \sum_I a_I v_I$ has rank ϕ_ω at least $n - r$ by the rank-nulity formula, with equality if and only if ω is decomposable. The matrix of ϕ_ω involves elements of the ring generated by a_I, and the condition to be rank = n-r is that all larger minors vanish, giving a collection of polynomials in a_I with simultaneous zero set precisely the coordinates of the decomposable elements.