Mathematics 552 Algebra II Spring 2017

The expression of the determinant of a square matrix in terms of products of determinants of submatrices

Jacobson covers Laplace's Formula on page 416, volume 1. It can be generalized slightly to computing minors of a matrix in terms of products of minors of smaller matrices.

Let M and N be free K-modules for a commutative ring K, with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively. For each k-element list I of integers $1 \le i_1 < i_2 \cdots, i_k \le m$ let $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then a basis for $\Lambda^k(M)$ is given by the v_I as I runs over all such lists of k integers chosen from the interval $[1, m]$.

Any K-homomorphism $\phi : M \to N$ is described by computing the image of basis vectors : $\phi(v_i) = \sum_{j=1}^n a_{ij} w_j$, giving an isomorphism of $Hom_K(M, N)$ with the K-module of $m \times n$ matrices. When P is a third K-module with basis $z_1, \ldots, z_p$ the composition of linear maps $M \to N \to P$ transfers to composition law taking an $m \times n$ matrix and an $n \times p$ matrix to an $m \times p$ matrix which gives the usual multiplication law for matrices.

The matrix associated to $\Lambda^k(\phi)$ is given by computing $\Lambda^k(\phi)(v_I) = \sum_J a_{IJ} w_J$ where I is an increasing list of k-integers chosen in the interval $[1, m]$ and J runs over all increasing lists of k-integers chosen in the interval $[1, n]$. If $\Lambda^l(\phi)(v_R) = \sum_S a_{RS} w_S$ and $\Lambda^{k+l}(\phi)(v_T) = \sum_U a_{TU} w_U$ we can compute $\Lambda^{k+l}(\phi)(v_I \wedge v_R)$ in two ways. The wedge product $w_J \wedge w_S = 0$ unless J and S are disjoint, in which case $w_J \wedge w_S = \epsilon_{JS} w_{J \cup S}$ where $\epsilon_{JS} = \pm 1$ by the alternating property of the wedge product. When J is the 1-element set containing j and S is the complement in $\{1, \ldots, n\}$ then $\epsilon_{JS} = (-1)^{j-1}$.

For disjoint I, R lists of length k and l respectively with $I \cup R = T$

$$a_{TU} \epsilon_{IR} = \sum_{J,S} a_{IJ} a_{RS} \epsilon_{JS}$$

where the sum is taken over all choices of disjoint lists of increasing integers from $[1, n]$ of length k and l respectively such that $J \cup S = U$. This follows by computing

$$\Lambda^{k+l}(\phi)(v_I \wedge v_T) = \Lambda^k(v_I) \wedge \Lambda^l(v_R)$$

and comparing coefficients of basis vectors on each side.

As mentioned in the writeup on the Cayley-Binet formula the coefficients a_{IJ} are the determinants of the submatrix of all elements a_{ij} with $i \in I, j \in J$. So the formula expresses determinants of submatrices of A in terms of determinants of smaller submatrices.

Example 1: $k = 1, l = m - 1, M = N$

In this case a_{ij} form a square matrix of size m. Let I_j be the complement of j in $\{1, \ldots, m\}$. The formula above gives

$$(-1)^{i-1} det(A) = \sum_j (-1)^{j-1} a_{ij} a_{I_i I_j}$$

which is the usual formula for the determinant of a square matrix in terms of the i-th row and determinants of smaller matrices. Since the determinant defined by exterior algebra agrees with the usual one for 1×1 matrices, this shows that they agree in general.

Example 2: $l = m - k, M = N$

Let I' be the complement of I in $\{1, \ldots, m\}$. Choose a fixed I of size k. The formula above then reads

$$det(A) = \sum_J a_{IJ} a_{I'J'} \epsilon_{II'} \epsilon_{JJ'}$$

Example 3: $k = 2, l = m - 2, m = 4, M = N$ Apply the previous example to the matrix

$$\begin{pmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \\ 0 & a2 & a3 & a4 \\ 0 & b2 & b3 & b4 \end{pmatrix}$$

taking $I = \{3, 4\}$ and noting that $a_{IJ} = 0$ if $1 \in J$ due to the zeros in the first column of rows 3 and 4. Further $a_{IJ} = a_t b_u - a_u b_t$ for $J = \{t, u\} = \{2, 3\}, \{2, 4\}, \{3, 4\}$ We obtain $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$ where $p_{ij} = a_i b_j - a_j b_i$, a relation among the minors of the two dimensional matrix

$$\begin{pmatrix} a1 & a2 & a3 & a4 \\ b1 & b2 & b3 & b4 \end{pmatrix}$$