Mathematics 552 **Algebra II** **Spring 2017**

Week 7 Wrapup of Semisimple rings and Representation Theory
Jacobson II: 5.1-5.3. 5.5, 5.6

1. Let G be a finite group and consider representations of G on finite dimensional complex vector spaces.

   a) Let V, W be irreducible representations of G. Show that the decomposition of $V^* \otimes W$ into irreducible representations contains exactly 1 copy of the trivial representation of G if V, W are isomorphic, and 0 otherwise (see previously assigned Jacobson II 5.2.7).

   b) Show that the element $t = \sum_{g \in G} g$ in $\mathbf{C}[G]$ is in the center of the group algebra, hence by Schur's lemma acts by a scalar on any irreducible representation V. Show that if the subspace $W = tV \subset V$ is nonzero, then V is the trivial representation. Show that $\chi_V(t) = 0$ if V is not the trivial representation, and $\chi_V(t) = |G|$ if V is the trivial representation.

   c) Show using a),b) that if V, W are irreducible representations then $\chi_{V*\otimes W}(t)$ is 0 or $|G|$ according as V, W are not isomorphic or are isomorphic. Deduce that the characters of irreducible representations of G form an orthonormal family of functions with respect to the pairing $< \phi, \psi >= (1/|G|) \sum_{g \in G} \phi(g^{-1})\psi(g)$.

   d) Show that if a representation V of G decomposes as a sum of irreducibles $V = \oplus m_i V_i$ with V_i, V_j not isomorphic when $i \neq j$, then $< \chi_V, \chi_V >= \sum m_i^2$. Show that V is irreducible if and only if $< \chi_V, \chi_V >= 1$.

   e) Show that if $W = \oplus m_i V_i$ is the decomposition of W into distinct irreducibles, then $< \chi_W, \chi_{V_i} >= m_i$. This, together with d) allows one to check from characters if a representation is irreducible, and if so how many times it appears in another representation.

2. The complex representations of S_4, the symmetric group on four letters.

   a) Find the conjugacy classes in S_4, find all normal subgroups, and determine how many irreducible complex representations S_4 has.

   b) Show that the group G of rotations preserving a cube permutes the 4 diagonals joining opposite corners of the cube and that this gives an injective group homomorphism from G to S_4. Show that G acts transitively on the 8 vertices of the cube and compute the stabilizer in G of a vertex. Show that the group G of rotations is isomorphic to S_4

c) Show that the group G of rotations of the cube maps onto the group of permutations of the lines joining centers of opposite faces and that S_3 is a quotient of G. Use this to find some irreducible representations of S_4 and to determine the dimension of the remaining irreducible representations.

d) Show that the natural representation of G as linear maps of R^3 is irreducible

e) Let W be the representation of G on the vector space of complex functions on the faces of the cube. Compute the character of W and express W as a sum of irreducible representations.

3. Let G be a subgroup of the group $GL(n, \mathbf{C})$ of invertible $n \times n$ complex matrices. Suppose that $\sum_{g \in G} trace(g) = 0$. Show that $\sum_{g \in G} g = 0$.

4. Jacobson II 5.5.7

5. Jacobson II 5.6.1

6. Jacobson II 5.6.4