§ 1. What is Algebra?

What is algebra? Is it a branch of mathematics, a method or a frame of mind? Such questions do not of course admit either short or unambiguous answers. One can attempt a description of the place occupied by algebra in mathematics by drawing attention to the process for which Hermann Weyl coined the unpronounceable word 'coordinatisation' (see [H. Weyl **109** (1939), Chap. I, § 4]). An individual might find his way about the world relying exclusively on his sense organs, sight, feeling, on his experience of manipulating objects in the world outside and on the intuition resulting from this. However, there is another possible approach: by means of *measurements*, subjective impressions can be transformed into objective marks, into numbers, which are then capable of being preserved indefinitely, of being communicated to other individuals who have not experienced the same impressions, and most importantly, which can be operated on to provide new information concerning the objects of the measurement.

The oldest example is the idea of *counting* (coordinatisation) and *calculation* (operation), which allow us to draw conclusions on the number of objects without handling them all at once. Attempts to 'measure' or to 'express as a number' a variety of objects gave rise to fractions and negative numbers in addition to the whole numbers. The attempt to express the diagonal of a square of side 1 as a number led to a famous crisis of the mathematics of early antiquity and to the construction of irrational numbers.

Measurement determines the points of a line by real numbers, and much more widely, expresses many physical quantities as numbers. To Galileo is due the most extreme statement in his time of the idea of coordinatisation: 'Measure everything that is measurable, and make measurable everything that is not yet so'. The success of this idea, starting from the time of Galileo, was brilliant. The creation of analytic geometry allowed us to represent points of the plane by pairs of numbers, and points of space by triples, and by means of operations with numbers, led to the discovery of ever new geometric facts. However, the success of analytic geometry is mainly based on the fact that it reduces to numbers not only points, but also curves, surfaces and so on. For example, a curve in the plane is given by an equation $F(x, y) = 0$; in the case of a line, F is a linear polynomial, and is determined by its 3 coefficients: the coefficients of x and y and the constant term. In the case of a conic section we have a curve of degree 2, determined by its 6 coefficients. If F is a polynomial of degree n then it is easy to see that it has $\frac{1}{2}(n+1)(n+2)$ coefficients; the corresponding curve is determined by these coefficients in the same way that a point is given by its coordinates.

In order to express as numbers the roots of an equation, the complex numbers were introduced, and this takes a step into a completely new branch of mathematics, which includes elliptic functions and Riemann surfaces.

For a long time it might have seemed that the path indicated by Galileo consisted of measuring 'everything' in terms of a known and undisputed collection of numbers, and that the problem consists just of creating more and more subtle methods of measurements, such as Cartesian coordinates or new physical instruments. Admittedly, from time to time the numbers considered as known (or simply called numbers) turned out to be inadequate: this led to a 'crisis', which had to be resolved by extending the notion of number, creating a new form of numbers, which themselves soon came to be considered as the unique possibility. In any case, as a rule, at any given moment the notion of number was considered to be completely clear, and the development moved only in the direction of extending it:

$$\text{`1, 2, many'} \Rightarrow \text{natural numbers} \Rightarrow \text{integers}$$
$$\Rightarrow \text{rationals} \Rightarrow \text{reals} \Rightarrow \text{complex numbers.}$$

But matrixes, for example, form a completely independent world of 'number-like objects', which cannot be included in this chain. Simultaneously with them, quaternions were discovered, and then other 'hypercomplex systems' (now called algebras). Infinitesimal transformations led to differential operators, for which the natural operation turns out to be something completely new, the Poisson bracket. Finite fields turned up in algebra, and p-adic numbers in number theory. Gradually, it became clear that the attempt to find a unified all-embracing concept of number is absolutely hopeless. In this situation the principle declared by Galileo could be accused of intolerance; for the requirement to 'make measurable *everything* which is not yet so' clearly discriminates against anything which stubbornly refuses to be measurable, excluding it from the sphere of interest of science, and possibly even of reason (and thus becomes a *secondary quality* or *secunda causa* in the terminology of Galileo). Even if, more modestly, the polemic term 'everything' is restricted to objects of physics and mathematics, more and more of these turned up which could not be 'measured' in terms of 'ordinary numbers'.

The principle of coordinatisation can nevertheless be preserved, provided we admit that the set of 'number-like objects' by means of which coordinatisation is achieved can be just as diverse as the world of physical and mathematical objects they coordinatise. The objects which serve as 'coordinates' should satisfy only certain conditions of a very general character.

They must be individually distinguishable. For example, whereas all points of a line have identical properties (the line is homogeneous), and a point can only be fixed by putting a finger on it, numbers are all individual: $3, 7/2, \sqrt{2}, \pi$ and so on. (The same principle is applied when newborn puppies, indistinguishable to the owner, have different coloured ribbons tied round their necks to distinguish them.)

They should be sufficiently abstract to reflect properties common to a wide circle of phenomenons.

Certain fundamental aspects of the situations under study should be reflected in *operations* that can be carried out on the objects being coordinatised: addition, multiplication, comparison of magnitudes, differentiation, forming Poisson brackets and so on.

We can now formulate the point we are making in more detail, as follows:

Thesis. *Anything which is the object of mathematical study (curves and surfaces, maps, symmetries, crystals, quantum mechanical quantities and so on) can be 'coordinatised' or 'measured'. However, for such a coordinatisation the 'ordinary' numbers are by no means adequate.*

Conversely, when we meet a new type of object, we are forced to construct (or to discover) new types of 'quantities' to coordinatise them. The construction and the study of the quantities arising in this way is what characterises the place of algebra in mathematics (of course, very approximately).

From this point of view, the development of any branch of algebra consists of two stages. The first of these is the birth of the new type of algebraic objects out of some problem of coordinatisation. The second is their subsequent career, that is, the systematic development of the theory of this class of objects; this is sometimes closely related, and sometimes almost completely unrelated to the area in connection with which the objects arose. In what follows we will try not to lose sight of these two stages. But since algebra courses are often exclusively concerned with the second stage, we will maintain the balance by paying a little more attention to the first.

We conclude this section with two examples of coordinatisation which are somewhat less standard than those considered up to now.

Example 1. The Dictionary of Quantum Mechanics. In quantum mechanics, the basic physical notions are 'coordinatised' by mathematical objects, as follows.

Physical notion	Mathematical notion
State of a physical system	Line φ in an ∞-dimensional complex Hilbert space
Scalar physical quantity	Self-adjoint operator
Simultaneously measurable quantities	Commuting operators
Quantity taking a precise value λ in a state φ	Operator having φ as eigenvector with eigenvalue λ
Set of values of quantities obtainable by measurement	Spectrum of an operator
Probability of transition from state φ to state ψ	$\lvert(\varphi, \psi)\rvert$, where $\lvert\varphi\rvert = \lvert\psi\rvert = 1$

Example 2. Finite Models for Systems of Incidence and Parallelism Axioms. We start with a small digression. In the axiomatic construction of geometry, we often consider not the whole set of axioms, but just some part of them; to be

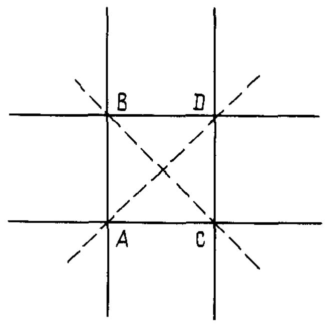

Fig. 1

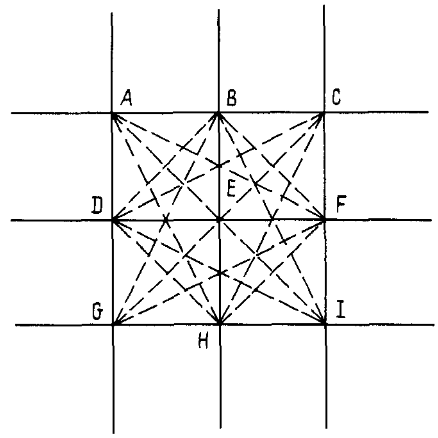

Fig. 2

concrete we only discuss plane geometry here. The question then arises as to what realisations of the chosen set of axioms are possible: do there exists other systems of objects, apart from 'ordinary' plane geometry, for which the set of axioms is satisfied? We consider now a very natural set of axioms of 'incidence and parallelism'.

(a) Through any two distinct points there is one and only one line.

(b) Given any line and a point not on it, there exists one and only one other line through the point and not intersecting the line (that is, parallel to it).

(c) There exist three points not on any line.

It turns out that this set of axioms admits many realisations, including some which, in stark contrast to our intuition, have only a finite number of points and lines. Two such realisations are depicted in Figures 1 and 2. The model of Figure 1 has 4 points A, B, C, D and 6 lines AB, CD; AD, BC; AC, BD. That of Figure 2 has 9 points, A, B, C, D, E, F, G, H, I and 12 lines ABC, DEF, GHI; ADG, BEH, CFI; AEI, BFG, CDH; CEG, BDI, AFH. The reader can easily verify that axioms (a), (b), (c) are satisfied; in our list of lines, the families of parallel lines are separated by semicolons.

We return to our main theme, and attempt to 'coordinatise' the model of axioms (a), (b), (c) just constructed. For the first of these we use the following construction: write $\mathbb{0}$ and $\mathbb{1}$ for the property of an integer being even or odd respectively; then define operations on the symbols $\mathbb{0}$ and $\mathbb{1}$ by analogy with the way in which the corresponding properties of integers behave under addition and multiplication. For example, since the sum of an even and an odd integer is odd, we write $\mathbb{0} + \mathbb{1} = \mathbb{1}$, and so on. The result can be expressed in the 'addition and multiplication tables' of Figures 3 and 4.

The pair of quantities $\mathbb{0}$ and $\mathbb{1}$ with the operations defined on them as above serve us in coordinatising the 'geometry' of Figure 1. For this, we give points coordinates (X, Y) as follows:

+	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{0}$	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{0}$

Fig. 3

×	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{0}$	$\mathbb{0}$	$\mathbb{0}$
$\mathbb{1}$	$\mathbb{0}$	$\mathbb{1}$

Fig. 4

$$A = (\mathbb{0}, \mathbb{0}), \quad B = (\mathbb{0}, \mathbb{1}), \quad C = (\mathbb{1}, \mathbb{0}), \quad D = (\mathbb{1}, \mathbb{1}).$$

It is easy to check that the lines of the geometry are then defined by the linear equations:

$$AB\colon \mathbb{1}X = \mathbb{0}; \quad CD\colon \mathbb{1}X = \mathbb{1}; \quad AD\colon \mathbb{1}X + \mathbb{1}Y = \mathbb{0};$$

$$BC\colon \mathbb{1}X + \mathbb{1}Y = \mathbb{1}; \quad AC\colon \mathbb{1}Y = \mathbb{0}; \quad BD\colon \mathbb{1}Y = \mathbb{1}.$$

In fact these are the only 6 nontrivial linear equations which can be formed using the two quantities $\mathbb{0}$ and $\mathbb{1}$.

The construction for the geometry of Figure 2 is similar, but slightly more complicated: suppose that we divide up all integers into 3 sets U, V and W as follows:

$$U = \text{integers divisible by 3,}$$

$$V = \text{integers with remainder 1 on dividing by 3,}$$

$$W = \text{integers with remainder 2 on dividing by 3.}$$

The operations on the symbols U, V, W is defined as in the first example; for example, a number in V plus a number in W always gives a number in U, and so we set $V + W = U$; similarly, the product of two numbers in W is always a number in V, so we set $W \cdot W = V$. The reader can easily write out the corresponding addition and multiplication tables.

It is then easy to check that the geometry of Figure 2 is coordinatised by our quantities U, V, W as follows: the points are

$$A = (U, U), \quad B = (U, V), \quad C = (U, W), \quad D = (V, U) \quad E = (V, V),$$
$$F = (V, W), \quad G = (W, U), \quad H = (W, V), \quad I = (W, W);$$

and the lines are again given by all possible linear equations which can be written out using the three symbols U, V, W; for example, AFH is given by $VX + VY = U$, and DCH by $VX + WY = V$.

Thus we have constructed finite number systems in order to coordinatise finite geometries. We will return to the discussion of these constructions later.

Already these few examples give an initial impression of what kind of objects can be used in one or other version of 'coordinatisation'. First of all, the collection of objects to be used must be rigorously delineated; in other words, we must indicate a set (or perhaps several sets) of which these objects can be elements. Secondly, we must be able to operate on the objects, that is, we must define *operations*, which from one or more elements of the set (or sets) allow us to construct new elements. For the moment, no further restrictions on the nature of the sets to be used are imposed; in the same way, an operation may be a completely arbitrary rule taking a set of k elements into a new element. All the same, these operations will usually preserve some similarities with operations on numbers. In particular, in all the situations we will discuss, $k = 1$ or 2. The basic examples of operations, with which all subsequent constructions should be compared, will be: the operation $a \mapsto -a$ taking any number to its negative; the operation $b \mapsto b^{-1}$ taking any nonzero number b to its inverse (for each of these $k = 1$); and the operations $(a, b) \mapsto a + b$ and ab of addition and multiplication (for each of these $k = 2$).

§ 2. Fields

We start by describing one type of 'sets with operations' as described in § 1 which corresponds most closely to our intuition of numbers.

A *field* is a set K on which two operations are defined, each taking two elements of K into a third; these operations are called *addition* and *multiplication*, and the result of applying them to elements a and b is denoted by $a + b$ and ab. The operations are required to satisfy the following conditions:

Addition:

Commutativity: $a + b = b + a$;

Associativity: $a + (b + c) = (a + b) + c$;

Existence of zero: there exists an element $0 \in K$ such that $a + 0 = a$ for every a (it can be shown that this element is unique);

Existence of negative: for any a there exists an element $(-a)$ such that $a + (-a) = 0$ (it can be shown that this element is unique).

Multiplication:

Commutativity: $ab = ba$;

Associativity: $a(bc) = (ab)c$;

Existence of unity: there exists an element $1 \in K$ such that $a1 = a$ for every a (it can be shown that this element is unique);

Existence of inverse: for any $a \neq 0$ there exists an element a^{-1} such that $aa^{-1} = 1$ (it can be shown that for given a, this element is unique).

Addition and multiplication:

Distributivity: $a(b + c) = ab + ac$.

Finally, we assume that a field does not consist only of the element 0, or equivalently, that $0 \neq 1$.