

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
January 2023

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Classify all groups of order 309, up to isomorphism.
2. Let A be the abelian group with generators x, y, z and the relations

$$4x + 3y + z = 0, \quad x + 2y + 3z = 0, \quad 3x + 2y + 5z = 0$$

Show that A is a cyclic abelian group, and determine its order.

3. Let A be a complex $n \times n$ matrix. Prove that there is an invertible complex $n \times n$ matrix B such that $AB = BA^t$. (A^t is the transpose of A .)

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Prove that the subring $\mathbb{Z}[3i]$ of $\mathbb{C}$ is not a Principal Ideal Domain (PID).

5. If $R = \mathbb{Z}[x]$, show that the sequence

$$R \xrightarrow{f} R^2 \xrightarrow{g} R$$

is exact, where $f(a) = (ax, -2a)$ and $g(c, d) = 2c + dx$.

End of Session on Algebra

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Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

1. $a, b > 0, a \neq b$. Find $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx$ by using residue calculus.
2. λ is purely imaginary. Prove that $z = \lambda - \frac{1}{3}e^{z^2}$ has exactly one solution in the strip $\mathbb{S} := \{z = x + iy : -1 \leq x \leq 1\}$.
3. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{A} := \{z \in \mathbb{C} : 0 < \arg z < \frac{2\pi}{5}\}$. Find an explicit holomorphic bijection $f : \mathbb{D} \rightarrow \mathbb{A}$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $\mathbb{S} := \{z = x + iy : -1 \leq x \leq 1\}$. $f : \mathbb{S} \rightarrow \mathbb{C}$ is a bounded continuous function that is holomorphic on the interior of the strip $\mathbb{S}$. For $-1 \leq x \leq 1$ let $M(x) := \sup_{y \in \mathbb{R}} |f(x + iy)|$.
1. Suppose $M(1), M(-1) \leq 1$. Prove that $|f(z)| \leq 1$ for any $z \in \mathbb{S}$.
 2. Suppose $M(1), M(-1)$ are arbitrary. Prove that $M(0)^2 \leq M(-1) \cdot M(1)$ by deducing it from part 1 of this problem.
5. In this problem you can use Picard theorem that states that the set of values of a non-constant entire function assumes either the whole $\mathbb{C}$ or $\mathbb{C}$ minus a point. Suppose f is an entire function that satisfies the equation $f(f(z)) = cf(z) + (1 - c)z$ for some fixed $c \in \mathbb{C}$, $c \neq 0, 1$. Prove that $f(z)$ is of the form $az + b$.

End of Session on Complex Variables and Advanced Calculus

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Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Let f be a function in $C^2(\mathbb{R}^2)$. Assume that f is periodic:

$$f(x+1, y) = f(x, y+1) = f(x, y) \quad \forall (x, y) \in \mathbb{R}^2.$$

Prove that

$$\int_0^1 \int_0^1 \det(D^2 f) dx dy = 0,$$

$$\text{where } D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

(**Hint:** Do it first under a stronger assumption that $f \in C^3(\mathbb{R}^2)$.)

2. Let $N \in \mathbb{N}$, the set of positive integers, and let $\{f_n\} \subset L^1(\mathbb{R}^N)$ be a sequence of functions satisfying

(a) $f_n \rightarrow f$ in $L^1(\mathbb{R}^N)$,

(b) $\{f_n\}$ is uniformly equicontinuous in $\mathbb{R}^N$, that is, for every $\epsilon > 0$ there exists a constant $\delta = \delta(\epsilon) > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \geq 1$, $x, y \in \mathbb{R}^N$, and $|x - y| < \delta$.

Prove that

$$\lim_{|x| \rightarrow \infty} \sup_{n \in \mathbb{N}} |f_n(x)| = 0.$$

3. Let $f \in L^1([0, 1])$, and let $\{f_n\}$ be a sequence of functions converging to f almost everywhere in $[0, 1]$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 \left| |f_n - f| - |f_n| + |f| \right| dx = 0.$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $\{f_n(x)\}$ and $f(x)$ be continuous functions on $[0, 1]$, and let g be a continuous function on $(-\infty, \infty)$. Assume that $\{f_n(x)\}$ converges to $f(x)$ in $[0, 1]$ in Lebesgue measure. Prove that $\{g(f_n(x))\}$ converges to $g(f(x))$ in $[0, 1]$ in Lebesgue measure.

5. Let $\{g_n\} \in C^1((0, 1))$, and $g \in C^2((0, 1))$. Assume that

$$\lim_{n \rightarrow \infty} \sup_{0 < x < 1} (|g_n(x) - g(x)| + |g'_n(x) - g'(x)|) = 0.$$

Prove that, for any sequence of positive numbers $\{h_n\}$ going to 0, and for any $0 < a < b < 1$, it holds

$$\lim_{n \rightarrow \infty} \frac{1}{(h_n)^2} \int_a^b [g_n(x + h_n) + g_n(x - h_n) - 2g_n(x)] dx = \int_a^b g''(x) dx.$$

End of Session on Real Variables and Elementary Point-Set Topology