

RUTGERS UNIVERSITY  
GRADUATE PROGRAM IN MATHEMATICS  
Written Qualifying Examination  
January 2017

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in the order that they appear in the book.

**Part I. Answer all questions.**

1. Prove that any complex square matrix is similar to its transpose matrix.
2. Prove that if the ring of polynomials  $R[x]$  over a commutative domain  $R$  with identity is a principal ideal ring then  $R$  is a field.
3. Prove that there are no simple groups of order 18.

**Part II. Answer one of the two questions.**

**If you work on both questions, indicate clearly which one should be graded.**

4. Prove that the groups  $D_6$  and  $A_4$  are not isomorphic. (*Here,  $D_6$  is the symmetry group of the hexagon and  $A_4$  is the alternating group of even permutations on 4 letters.*)
5. Let  $p$  be prime and let  $G$  be a  $p$ -group. Let  $X$  be a finite set with  $|X|$  not divisible by  $p$ . Suppose that  $G$  acts on  $X$ . Prove that there exists  $x \in X$  with orbit  $G \cdot x = \{x\}$ , that is, the action of  $G$  on  $X$  must have at least one fixed point.

**End of Session 1**

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**Session 2. Complex Variables and Advanced Calculus**

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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**Part I. Answer all questions.**

1. Given any polynomials  $z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0$  and  $a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ , prove that for all sufficiently large  $r > 0$ ,

$$\int_{|z|=r} \frac{a_{n-1}z^{n-1} + \cdots + a_1z + a_0}{z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0} dz = 2\pi a_{n-1}i.$$

2. Find a conformal map from the upper half-disc  $D = \{z = x + iy : |z| < 1 \text{ and } y > 0\}$  to the first quadrant  $Q = \{w = u + iv : u > 0 \text{ and } v > 0\}$ . Furthermore, describe all such maps.
3. Suppose  $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ , i.e., continuous and Lebesgue integrable on  $\mathbb{R}^+$ . Define  $F(z) = \int_0^\infty f(t)e^{izt} dt$  for  $z \in \overline{\mathbb{H}}$ , the closure of the upper half plane  $\mathbb{H}$ .
- (a). Prove that  $F(z)$  defines a bounded, continuous function on  $\overline{\mathbb{H}}$ , and is holomorphic in  $\mathbb{H}$ . Prove further that  $F(z) \rightarrow 0$  as  $\Im z \rightarrow \infty$ .
- (b). Assume that  $\int_0^\infty f(t)e^{ixt} dt = 0$  for all  $x \in \mathbb{R}$ . Prove that  $F(z) \equiv 0$  for all  $z \in \overline{\mathbb{H}}$ .

**Part II. Answer one of the two questions.**

**If you work on both questions, indicate clearly which one should be graded.**

4. Suppose that  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  converges and  $|f(z)| \leq 1$  for all  $z \in \mathbb{C}$  with  $|z| < 1$ . Prove that  $|a_0|^2 + |a_1| \leq 1$ . In addition, determine all  $f$  that satisfies  $|a_0|^2 + |a_1| = 1$ .
5. Let  $f$  be a nonconstant meromorphic function on the complex plane such that  $f(z+1) = f(z)$  and  $f(z+i) = f(z)$  for every  $z \in \mathbb{C}$ .
- (a). Prove that in the square  $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1, 0 \leq \operatorname{Im} z < 1\}$  the function  $f$  must have either
- two (or more) different poles, or else
  - one (or more) single pole of order  $\geq 2$ .
- (b). Prove that  $f$  must have at least two zeros (counting multiplicity) in the square  $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1, 0 \leq \operatorname{Im} z < 1\}$ .

**End of Session 2**

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**Session 3. Real Variables and Elementary Point-Set Topology**

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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**Part I. Answer all questions.**

1. The sum  $A + B$  of two subsets of  $\mathbb{R}^n$  is  $A + B = \{x + y; x \in A, y \in B\}$ .
- a) Show that if  $A$  is closed and  $B$  is compact, then  $A + B$  is closed.
  - b) Show that the sum  $A + B$  of two compact subsets of  $\mathbb{R}^n$  is compact.
  - c) Show that the sum of two closed sets is not necessarily closed.
2. For any  $a > 0$ , show that  $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$  converges absolutely and uniformly to a continuous function on  $[0, a]$ . Show that

$$\int_0^x f(y) dy = \sum_{n=0}^{\infty} \int_0^x \frac{\sin(ny)}{n^2} dy$$

and that the right-hand side converges uniformly to the the left-hand side on  $[0, a]$ .

3. a) Let  $m$  denote Lebesgue measure on the bounded interval  $[a, b]$ . Show that if  $\{f_n; n \geq 1\}$  is a sequence of real-valued, Lebesgue measurable functions on  $[a, b]$  and  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  almost everywhere with respect to Lebesgue measure, then  $f_n$  converges to  $f$  in (Lebesgue) measure.
- b) Show that in general, convergence of a sequence of functions in measure on a finite interval  $[a, b]$  does not imply convergence almost everywhere.

**Part II. Answer one of the two questions.**

**If you work on both questions, indicate clearly which one should be graded.**

4. Let  $a, b$  be real numbers such that  $a < b$ .
- (i). Define what it means for a function  $f : [a, b] \mapsto \mathbb{C}$  to be "absolutely continuous".

- (ii). **Prove**, using the definition of absolute continuity given in Part 1, that the product of two absolutely continuous functions is absolutely continuous.

5. In this problem,  $m_2$  is the two-dimensional Lebesgue measure. We want to study the double integral

$$I(T) = \int_0^\infty \int_0^T e^{-xy} \sin x \, dx \, dy,$$

that is,

$$I(T) = \int_{\mathbb{R} \times \mathbb{R}} e^{-xy} \sin x \chi_{E(T)}(x, y) dm_2(x, y), \quad (1)$$

where  $\chi_{E(T)}$  is the characteristic function of the set

$$E(T) = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq T \text{ and } y \geq 0\}.$$

- (i). **Prove** that the integrand of (1) (that is, the function  $f_T : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  given by

$$f_T(x, y) = e^{-xy} \sin x \chi_{E(T)}(x, y)$$

for  $(x, y) \in \mathbb{R} \times \mathbb{R}$  is integrable. (HINT:  $|\sin x| \leq |x|$ .)

- (ii). By computing  $I(T)$  in two different ways, **prove** that

$$\lim_{T \rightarrow +\infty} \int_0^T \frac{\sin x}{x} dx = \frac{\pi}{2},$$

**Justify all the steps rigorously.**

NOTE: You are allowed to use the facts that

$$\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2},$$

and

$$\int e^{-ax} \sin(x) \, dx = \frac{-e^{-ax}(\cos(x) + a \sin(x))}{1+a^2} + C.$$

**End of Session 3**