

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
August 2024

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1 (10 pts). Let G be any group of order $2024 = 2^3 \cdot 11 \cdot 23$. Show that G is solvable. (**Caution:** Do not use Burnside's $p^a q^b$ -theorem.)

2 (10 pts). Let T and S be two linear transformations $\mathbb{C}^5 \rightarrow \mathbb{C}^5$, with the same characteristic polynomial $x^2(x-2)(x^2-9)$. Suppose that the kernels of T and of S have the same dimension. Are T and S similar? Justify your answer.

3. Let R be a PID and $I \triangleleft R$ a nonzero ideal of R .

(a) **(8 pts)** Prove that there are only finitely many ideals of R that contain I .

(b) **(2 pts)** Give an example to show that the statement (a) is false if R is a UFD.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let G be a subgroup of the multiplicative group of nonzero elements of a field $\mathbb{F}$.

(a) **(8 pts)** Suppose G is finite. Prove that G is cyclic.

(b) **(2 pts)** Is the conclusion in (a) still true if G is infinite countable? Justify your answer.

5. Let $\mathbb{F}$ be an infinite field and V a vector space over $\mathbb{F}$.

(a) **(8 pts)** Prove that V cannot be written as a finite union of proper subspaces of V . That is, if $V = \bigcup_{i=1}^n V_i$ for some $n \in \mathbb{Z}_{\geq 1}$ and some subspaces $V_1, \dots, V_n$ of V , then $V_i = V$ for some $1 \leq i \leq n$.

(b) **(2 pts)** Show that the statement in (a) is false if $\mathbb{F}$ is a finite field and $\dim_{\mathbb{F}} V = m$ for some $m \in \mathbb{Z}_{\geq 2}$.

End of Session on Algebra

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Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. For any $|a| < 1$, calculate the following integral:

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2}.$$

2. Let $u(x, y) = \log(x^2 + (1 - y)^2)$. Find a function $v(x, y)$ such that $f(z) = f(x + \sqrt{-1}y) = u(x, y) + \sqrt{-1}v(x, y)$ is a holomorphic function on the open set $D = \mathbb{C} \setminus \{(0, y) : y \leq 1\}$. Then find the image set $f(D)$.

3. Let $f(z)$ be a holomorphic function in $|z| < 1$ and satisfies $f(0) = 0$. Prove that the series $\sum_{n=1}^{+\infty} f(z^n)$ is convergent and defines a holomorphic function in $|z| < 1$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $P_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be any polynomial with leading coefficient 1. Prove that

$$\max_{|z| \leq 1} |P_n(z)| \geq 1.$$

5. Find a conformal map that maps one-third of the unit disk $D_1 = \{z \in \mathbb{C} : |z| < 1, 0 < \text{Arg}(z) < 2\pi/3\}$ to the upper half plane.

End of Session on Complex Variables and Advanced Calculus

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Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. Let (X, ρ) be a metric space.

- (a) Suppose that (X, ρ) is separable. Prove that if Y is any subset of X , then (Y, ρ) is separable.
- (b) Suppose that (X, ρ) is compact. Let $\mathcal{F}$ be any set of real valued functions on X that is uniformly bounded and equicontinuous. Is the function

$$g(x) = \sup\{ f(x) : f \in \mathcal{F} \}$$

necessarily continuous? Prove that your answer is correct.

2. Let μ be Lebesgue measure on $\mathbb{R}$.

- (a) Let

$$A := \left\{ f \in L^1(\mu) : \int_{\mathbb{R}} |f(x)|^2 d\mu \geq 1 \right\} .$$

Is A a closed subset of $L^1(\mu)$? Prove that your answer is correct.

- (b) Let

$$B := \left\{ f \in L^1(\mu) : \int_{\mathbb{R}} |f(x)|^2 d\mu \leq 1 \right\} .$$

Is B a closed subset of $L^1(\mu)$? Prove that your answer is correct.

3. Let μ denote Lebesgue measure on $[0, 1]$. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_{[0,1]} |f_n(x)|^2 d\mu(x) \leq 1 \quad \text{for all } n.$$

Suppose that the sequence $\{f_n\}$ converges to zero in measure. Show that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n(x) d\mu(x) = 0 .$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let μ^* denote Lebesgue outer measure on $\mathbb{R}$. Let A and B be any two subsets of $\mathbb{R}$ that are separated by a positive distance d . That is, if $x \in A$ and $y \in B$, then $|x - y| \geq d > 0$. Show that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) .$$

5: Let μ be Lebesgue measure on $\mathbb{R}$. For any $f \in L^1(\mu)$ and any $a > 0$, define $f_a(x) := af(ax)$.

(a) Prove that for all $a > 0$, $\int_{\mathbb{R}} f_a(x) d\mu = \int_{\mathbb{R}} f(x) d\mu$.

(b) Prove that

$$\lim_{a \rightarrow 1} \int_{\mathbb{R}} |f(x) - f_a(x)| d\mu = 0 .$$

End of Session on Real Variables and Elementary Point-Set Topology