RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2025

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let G be a group of order p^3, where p is a prime. Show that G contains either p^3 or $p^2 + p - 1$ conjugacy classes.

2. Let M be a finitely generated $\mathbb{Z}$-module such that $p^n M = 0$ holds for some prime number p and $n \geqslant 1$. Assume that $x \in M$ has annihilator $\mathrm{Ann}(x) = \{a \in \mathbb{Z} \mid ax = 0\} = \mathbb{Z}p^n$. Prove that $\mathbb{Z}x$ is a direct summand of M, that is, we have $M = \mathbb{Z}x \oplus M'$ for some submodule $M' \subset M$.

3. (a) Let A be an $n \times n$ matrix over $\mathbb{Q}$. Assume that $A^p = I \neq A$, where p is a prime and I is the identity matrix. Prove that $n \geqslant p - 1$.

(b) Give an example of a 2×2 matrix over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ such that $A^p = I \neq A$.

You may assume without proof that the polynomial $x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Z}$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let A be a commutative ring with multiplicative unit 1, and let $\phi : A \to A$ be a surjective ring homomorphism.

(a) Prove that if A is Noetherian, then ϕ is an isomorphism.

(b) Give an example where A is not Noetherian and ϕ is not injective.

5. Prove that $\mathbb{Z}[\sqrt{-2}] = \mathbb{Z}[x]/\langle x^2 + 2\rangle$ is a Euclidean domain.

End of Session on Algebra

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2025

Session 1. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Use the calculus of residues to show that

$$\int_0^\infty \frac{\log x}{(x^2+1)^2} dx = -\frac{\pi}{4}.$$

2. Let $p(z)$ be a polynomial with complex coefficients. Assume that the set $\{z \in \mathbb{C} : |p(z)| = 1\}$ is a disjoint union of ten simple closed curves. Prove that $\deg p \geqslant 10$.

3. **(A)** Show that the following function

$$F(z) = \int_0^1 e^{-t} t^{z-1} dt$$

is holomorphic in the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

(B) Expanding e^{-t} in a power series show that $F(z)$ extends to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at $z = 0, -1, -2, -3, \ldots$. Moreover, the residue of F at $z = -n$ is $\frac{(-1)^n}{n!}$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and suppose that there exist $\rho > 0$ and constants $A, B > 0$ such that

$$|f(z)| \leqslant A e^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C}.$$

Let $\mathfrak{n}_f(r)$ be the number of zeros of f (counted with their multiplicities) inside the disc with radius $r > 0$. Prove that the following holds:

(A) $\mathfrak{n}_f(r) \leqslant Cr^\rho$ for some $C > 0$ and all sufficiently large $r > 0$.

(B) If $z_1, z_2, \ldots$ denote the zeros of f, with $z_k \neq 0$, then for all $s > \rho$, we have

$$\sum_{k=1}^\infty \frac{1}{|z_k|^s} < \infty.$$

5. (A) Let f be an analytic isomorphism on $D = \{z \in \mathbb{C} : |z| < 1\}$, and let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad \text{where} \qquad a_n \in \mathbb{C},$$

be the power series representation of f. Prove that

$$\text{area}(f(D)) = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

(B) Let f be an analytic isomorphism on $\overline{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$, such that

$$|f(z)| \geqslant 1 \qquad \text{for} \qquad |z| = 1,$$

and $f(0) = 0$. Prove that the area of $f(D)$ satisfies

$$\text{area}(f(D)) \geqslant \pi.$$

End of Session 1

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Consider the function the function $f : (0, \infty) \to \mathbb{R}$ defined as

$$f(x) := \frac{\sin(x)}{x}.$$

(a) Is f in $L^1((0, \infty))$?

(b) Is f in $L^2((0, \infty))$?

(c) Define the sequence

$$a_N = \int_0^N f(x)dx, \qquad N \in \mathbb{N}.$$

Prove that $\lim_{n\to\infty} a_N$ exists.

2. Consider a sequence of Lebesgue measurable functions $\{f_n\}_n$ and suppose $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $\|f_n\|_1 \leqslant M$. Assume that there is a function f such that $f_n(x) \to f(x)$ for almost every x.

(a) Prove that $\|f\|_1 \leqslant M$.

(b) Does f_n necessarily converge to f in L^1 norm? If so prove it, if not, give a counterexample.

(c) Suppose now that $\{f_n\}_n$ is a sequence of Lebesgue measurable functions such that $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $\|f_n\|_1 \leqslant M$. Does f_n necessarily have a subsequence that converges in L^1-norm? If so, prove it, otherwise provide a counterexample.

3. Consider the function f defined as: for $0 < x < 1$ and $0 < y < 1$,

$$f(x, y) = -\frac{\cos(3x - y)\cos(x + 2y)}{1 - xy}. \tag{1}$$

Consider the following integrals:

$$I_1 := \int_0^1 \left(\int_0^1 f(x,y)\ dy \right)\ dx, \quad I_2 := \int_0^1 \left(\int_0^1 f(x,y)\ dx \right)\ dy \tag{2}$$

$$I_3 := \int_{[0,1]^2} f(x,y)\ dx \times dy. \tag{3}$$

Which of these integrals are finite? Which are equal to each other?

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. We define the distance between two non-empty sets $A, B \subset \mathbb{R}^2$ by

$$d(A, B) := \inf_{a \in A, b \in B} \|a - b\| \tag{4}$$

where $\| \cdot \|$ is the standard Euclidean norm.

(a) Is it possible for there to be two closed sets A, B such that $A \cap B = \varnothing$ and that $d(A, B) = 0$? If so, provide an example, otherwise provide a proof that it is impossible.

(b) Let A, B be non-empty subsets of $\mathbb{R}^2$ with A closed and B compact. Prove that $d(A, B) > 0$. (*Hint*: Prove that there exist $a \in A$ and $b \in B$ such that $d(A, B) = \|a - b\|$.)

5. Let λ denote the Lebesgue measure on $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable. Given a Borel set $B \subset \mathbb{R}$, we define

$$\mu(B) := \lambda(f^{-1}(B)). \tag{5}$$

(a) Prove that μ is a measure.

(b) Prove that for any μ-integrable function $g : \mathbb{R} \to \mathbb{R}$,

$$\int g(y)\ d\mu(y) = \int g \circ f(x)\ d\lambda(x). \tag{6}$$

End of Session on Real Variables and Elementary Point-Set Topology