Math 311-Final Exam
December 2016

Problem 1 (15 points): Let $f(x) = x^2 + x - 5$. Prove by definition that $f(x)$ is uniformly continuous over any closed interval $[-R, R]$ with $R > 0$. Also prove that $f(x)$ is not uniformly continuous over $(-\infty, \infty)$.

Problem 2 (15 points): Let $f(x)$ be a function over $(0, 1]$. Assume that $f(x)$ is uniformly continuous over $(0, 1]$. Prove that $\lim_{x \to 0^+} f(x)$ exists.

Problem 3 (10 points): Suppose that $f(x)$ is continuous over $(-1,1)$. Assume that $\lim_{x\to 0} f(x) = 1$. Prove that there is a $\delta > 0$ such that $f(x) > 1/2$ for $x \in (-\delta, \delta)$.

Problem 4 (15 points): Let $f(x)$ be a differentiable function over $[0,1]$. Suppose that $f(0) = f(1) = 0$. Prove that $f'(x) - 2f(x)$ must have a zero inside $(0,1)$. Namely, there is a $c \in (0,1)$ such that $f'(c) - 2f(c) = 0$.

Problem 5 (15 points): (a). Let $f(x)$ be a decreasing function over (a, b). Show that $f'(x) \leq 0$. (b). Let $f(x)$ be defined by $-x+2x^2\cos(\frac{1}{x})$ for $x \neq 0$ and define $f(0) = 0$. Show that $f'(0) < 0$. However, for any $\delta > 0$, $f(x)$ is not a decreasing function over $(-\delta, \delta)$.

Problem 6 (15 points): (A). For any $a < b$, prove that the closed interval $[a, b]$ is compact. (B). Show that the open interval (a, b) is not compact.

Problem 7 (10 points): Suppose that $f(x)$ is differentiable over (a, b) except possibly at $c \in (a, b)$. Suppose that $f(x)$ is continuous over (a, b) and $\lim_{x \to c} f'(x)$ exists. Prove that $f'(c)$ also exists.

Problem 8 (10 points): (A). Suppose $f(x)$ is differentiable in $(a - \delta, b + \delta)$ for a certain $\delta > 0$. Assume that $f'(a) > 0$ and $f'(b) < 0$. Show that there is a point $c \in (a, b)$ such that $f'(c) = 0$. (B). Construct a function f over $(-2, 2)$ which can not be the derivative of any function defined over $(-2, 2)$.